INTERSECTIONS OF HECKE CORRESPONDENCES ON MODULAR CURVES

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Abstract. We compute the arithmetic intersections of Hecke correspondences on the product of integral model of modular curve $\mathcal{X}_0(N)$ and relate it to the derivatives of certain Siegel Eisenstein series when N is odd and squarefree. We prove this by establishing a precise identity between the arithmetic intersection numbers on the Rapoport–Zink space associated to $\mathcal{X}_0(N)^2$ and the derivatives of local representation densities of quadratic forms.

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Date: December 28, 2025.

1. Introduction

1.1. **Background.** Let $\mathbb{H} = \{\tau = x + iy : x, y \in \mathbb{R}, y \neq 0\}$ be the union of the upper and lower half plane. Let $Y_0(1)$ be the fine moduli stack of elliptic curves over $\mathbb{C}$, whose coarse moduli space $Y_0(1)_{\mathbb{C}}$ has $\mathbb{C}$-points given by the quotient $\mathrm{GL}_2(\mathbb{Z})\backslash\mathbb{H}$ and is isomorphic to $\operatorname{Spec}\mathbb{C}[j]$ by the elliptic modular function $j = j(\tau)$. Let $X_0(1)$ be the compactification of $Y_0(1)$, its associated coarse moduli space $X_0(1)_{\mathbb{C}}$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$.

For a positive integer m, the m-th Hecke correspondence $T(m)$ (resp. $T(m)_{\mathbb{C}}$) is a divisor on the product $Y_0(1)^2$ (resp. $Y_0(1)^2_{\mathbb{C}}$) which parameterizes degree m isogenies between elliptic curves. Let $\overline{T}(m)$ (resp. $\overline{T}(m)_{\mathbb{C}}$) be the closure of $T(m)$ (resp. $T(m)_{\mathbb{C}}$) in the compactification $X_0(1)^2$ (resp. $X_0(1)^2_{\mathbb{C}}$). Kronecker and Hurwitz showed that as a line bundle on $X_0(1)^2_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$,

$$\overline{T}(m) = \mathcal{O}(\sigma_1(m), \sigma_1(m)) := \mathrm{pr}_1^*\mathcal{O}(\sigma_1(m)) \otimes \mathrm{pr}_2^*\mathcal{O}(\sigma_1(m)),$$

where $\sigma_1(m) = \sum_{d|m} d$ and $\mathrm{pr}_1, \mathrm{pr}_2$ are the two projections from $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ to its two factors. Notice that $\{\sigma_1(m)\}_{m\geqslant 1}$ is the Fourier coefficients of the following non-holomorphic modular form, which is also a special value of a weight 2 Eisenstein series of genus 1 (cf. [BGHZ08, §2.3]),

$$G_2(\tau) = \frac{1}{8\pi y} - \frac{1}{24} + \sum_{m\geqslant 1} \sigma_1(m) q^m, \tau = x + iy \text{ with } y \gg 0 \text{ and } q = \exp(2\pi i\tau).$$

Hurwitz (cf. [GK93, Proposition 2.4]) also studied the intersection number of two divisors $T(m_1)$ and $T(m_2)$ on the affine plane $Y_0(1)^2_{\mathbb{C}} \simeq \operatorname{Spec}\mathbb{C}[j, j']$. The two divisors intersect properly if and only if $m_1 m_2$ is not a perfect square, and in this case the intersection $T(m_1) \cap T(m_2)$ parameterizes certain CM elliptic curves over $\mathbb{C}$. It turns out that the intersection number equals to the summation of certain Fourier coefficients of a weight 2 genus 2 Siegel Eisenstein series. This is a special case of the *geometric Siegel-Weil formula* and yields a beautiful geometric proof of the Hurwitz class number formula.

In this article, we consider the *arithmetic intersection* between Hecke correspondences. Let N be an odd square-free positive integer. Denote by $\mathcal{Y}_0(N)$ the moduli stack of elliptic curves with $\Gamma_0(N)$-level structure defined by Katz and Mazur [KM85], whose compactification is constructed by Česnavičius in [Č17]. The product space $\mathcal{X}_0(N)^2 := \mathcal{X}_0(N) \times_{\mathbb{Z}} \mathcal{X}_0(N)$ is normal but not regular when $N \neq 1$. We construct a regular integral model $\mathcal{M}_0(N)$ of $\mathcal{X}_0(N)^2$ by blowing up at certain supersingular points of $\mathcal{X}_0(N)^2$. We also define an "integral" Hecke correspondence $\mathcal{T}(m)$ on the model $\mathcal{M}_0(N)$. It will be a Cartier divisor on $\mathcal{M}_0(N)$ whose generic fiber over $\mathbb{C}$ equals the m-th Hecke correspondence on $X_0(N)$. We study the intersection between three correspondences $\{\mathcal{T}(m_i)\}_{i=1,2,3}$ on $\mathcal{M}_0(N)$. The computation of the intersection numbers will be reduced to the intersections of "local" Hecke correspondences on the Rapoport–Zink space associated to the model $\mathcal{M}_0(N)$. We established an identity:

$$\begin{pmatrix} \text{Intersection numbers on} \\ \text{Rapoport–Zink space} \end{pmatrix} = \begin{pmatrix} \text{Derivatives of local density polynomial} \\ \text{of quadratic lattices} \end{pmatrix} \quad (1)$$

This finally builds up the bridge connecting the intersection numbers of Hecke correspondences and the summation of derivatives of certain Fourier coefficients of a weight 2 genus 3 Siegel Eisenstein series, which is a special case of the non-degenerate term of the *arithmetic Siegel-Weil formula* and belongs to an influential program initiated by Kudla [Kud97, Kud04].

The key result is the identity (1), as the global result follows easily by p-adic uniformization of the supersingular locus of $\mathcal{M}_0(N)$. The identity of the type in (1) are called *Kudla–Rapoport conjecture*, which can be regarded as the nonarchimedean part of the arithmetic Siegel–Weil formula.

The arithmetic Siegel–Weil formula was first established in the works of Kudla, Rapoport and Yang [KRY99, KRY04, KRY06] for GSpin Shimura varieties associated to $\mathrm{GSpin}(n-1,2)$ where $n=1,2$. The archimedean part was proved by the work of Garcia–Sankaran [GS19] and Bruinier-Yang [BY20]. The work of Gross–Keating [GK93] complemented by the ARGOS volume [VGW$^+$07] established the nonarchimedean part for the case $n=3$ with hyperspecial level. The case for general n with hyperspecial level and odd p was proved by Li and Zhang [LZ22b].

In fact, the Kudla–Rapoport conjecture was originally formulated for unitary Rapoport–Zink space associated to $\mathrm{U}(n-1,1)$ with hyperspecial level [KR11]. Terstiege [Ter13a] solved the $n=3$ case. The work of Li–Zhang [LZ22a] solved the general case of the original Kudla–Rapoport conjecture. The work of Li–Liu and Yao [LL22, Yao24] established an invariant for the exotic smooth models over ramified primes.

In the unitary case, there are also many works about Shimura varieties with bad reduction. However, because of the bad reduction, even the formulation of the formula was not clear for a long time. For Krämer models over ramified primes, the works of Shi [Shi23] and He–Shi–Yang [HSY20] proved the $n=2$ case. A general formulation was proposed and the case $n=3$ was proved in [HSY23] and finally He–Li–Shi–Yang [HLSY23] proved the general case. Over unramified primes, Cho [Cho22] proposed a general conjecture for the maximal parahoric levels. In [CHZ23], Cho–He–Zhang showed that a formulation similar to that of [HSY23] is the same as the formulation of [Cho22] and reduced it to a strong version of Tate conjecture for certain Deligne–Lusztig varieties.

Moreover, the unitary case has seen a number of other recent and thrilling developments, including formulas for degenerate terms by Bruinier–Howard [BH21] and Chen [Che24] and the function field case by Feng–Yun–Zhang [FYZ24].

We refer to the left hand side of (1) the geometric side, while the right hand side the analytic side. The Rapoport–Zink space appearing on the geometric side is determined by the quadratic lattice appearing on the analytic side. The case $N=1$, which locally corresponds to the rank 4 self-dual lattice $(\mathrm{M}_2(\mathbb{Z}_p), \det)$, is the classical work of Gross and Keating [GK93] complemented by the ARGOS volume [VGW$^+$07]. The question is widely open if the quadratic lattice is not self-dual, or equivalently, the Rapoport–Zink space has some non-hyperspecial level structures. The works of Kudla–Rapoport [KR00], Sankaran–Shi–Yang [SSY23], and Zhu [Zhu25] proved the case when L is of rank 3 but not self-dual, which corresponds to $\mathrm{GSpin}(1,2)$ Shimura varieties.

Our work essentially settled the case when L is a specific rank 4 lattice that is not self-dual, see (4) in the next section for the precise definition of the lattice (the self-dual case is exactly the work of Gross–Keating [GK93]). In particular, this corresponds to $\mathrm{GSpin}(2,2)$-Shimura varieties. Since the GSpin Shimura varieties are not of PEL type, such problems are much more involved. In the current work, we make essential use of the concrete moduli interpretation of the Rapoport–Zink space which only exists for certain low dimensional special cases.

Before we introduce the main results, we briefly mention the new difficulties that arise compared with the Gross–Keating's case. Because of the existence of non-trivial level structure, both of the geometric side and analytic side become much more complicated. For example, the RZ space will no longer be regular. To compute the intersection number, we consider the blow-up of the original space, which lacks a nice moduli interpretation. Moreover, the special fiber of the RZ space will have more complicated components compared with the Gross–Keating's case, which causes a lot of troubles for the study of special cycles. For the analytic side, essentially we need to consider the Whittaker functional of some representation that is no longer unramified, which is significantly harder than the unramified case. When the level of the modular curve becomes deeper, we don't even know how to construct a suitable regular integral model of the product of modular curves so that we can compute the special divisors and intersection numbers.

1.2. **Main results.** We only state the local version of our main result. Let p be an odd prime. Let $\mathbb{F}$ be the algebraic closure of the finite field $\mathbb{F}_p$ and $W = W(\mathbb{F})$. We fix a supersingular p-divisible group $\mathbb{X}$ over $\mathbb{F}$ of dimension 1 and height 2. Let $\mathbb{B} = \mathrm{End}^\circ(\mathbb{X}) := \mathrm{End}(\mathbb{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ be the space of quasi-endomorphisms of $\mathbb{X}$, which is isometric to the unique quaternion division algebra over $\mathbb{Q}_p$. Let q be the quadratic form on $\mathbb{B}$ given by the reduced norm of $\mathbb{B}$. Let $x_0 \in \mathbb{B}$ be an element such that $\nu_p(q(x_0)) = 1$ where ν_p is the valuation so that $\nu_p(p) = 1$. We consider the following functor $\mathcal{N}(x_0)$: for a a W-scheme S such that p is locally nilpotent in $\mathcal{O}_S$, the set $\mathcal{N}(x_0)(S)$ consists of tuples

$$\left(X_1 \xrightarrow{\pi_1} X_1', (\rho_1, \rho_1')\right), \left(X_2 \xrightarrow{\pi_2} X_2', (\rho_2, \rho_2')\right),$$

where $\{X_i, X_i'\}_{i=1,2}$ are deformations of $\mathbb{X}$ to S via the height 0 quasi-isogenies ρ_i and ρ_i' from $\mathbb{X}$ to X_i and X_i', and π_1, π_2 are deformations of the quasi-isogeny x_0 under the framing morphisms ρ_i and ρ_i'. The functor $\mathcal{N}(x_0)$ is represented by a formal scheme formally of finite type over W.

Let

$$\left(X_1^{\mathrm{univ}} \xrightarrow{x_{0,1}^{\mathrm{univ}}} X_1'^{\mathrm{univ}}, (\rho_1^{\mathrm{univ}}, \rho_1'^{\mathrm{univ}})\right), \quad \left(X_2^{\mathrm{univ}} \xrightarrow{x_{0,2}^{\mathrm{univ}}} X_2'^{\mathrm{univ}}, (\rho_2^{\mathrm{univ}}, \rho_2'^{\mathrm{univ}})\right) \tag{2}$$

be the universal object over the formal scheme $\mathcal{N}(x_0)$. Let $x \in \mathbb{B}$ be a non-zero element and denote by x' the element $x_0 \cdot x \cdot x_0^{-1}$. We have the following commutative diagram

$$\begin{array}{ccc} X_1^{\mathrm{univ}} & \overset{x}{\dashrightarrow} & X_2^{\mathrm{univ}} \\ {\scriptstyle x_{0,1}^{\mathrm{univ}}}\downarrow & & \downarrow{\scriptstyle x_{0,2}^{\mathrm{univ}}} \\ X_1'^{\mathrm{univ}} & \overset{x'}{\dashrightarrow} & X_2'^{\mathrm{univ}}, \end{array}$$

where the dotted arrows below x and x' mean that they are quasi-isogenies. For a subset $H \subset \mathbb{B}$, define the *special* $\mathcal{Z}$*-cycle* $\mathcal{Z}_{\mathcal{N}(x_0)}(H) \subset \mathcal{N}(x_0)$ (resp. *special* $\mathcal{Y}$*-cycle* $\mathcal{Y}_{\mathcal{N}(x_0)}(H) \subset \mathcal{N}(x_0)$) to be the closed formal subscheme over which both x and x' (resp. $x_0 \cdot x$ and $x' \cdot x_0$) deform to isogenies for all $x \in H$ (cf. Definition 4.9.1). As we will see later (see §1.4.2), one should regard the special $\mathcal{Z}$-cycles as local analogues of the global Hecke correspondences we mentioned previously, and $\mathcal{Y}$-cycles as a dual version of this (see Remark 1.4.7).

The formal scheme $\mathcal{N}(x_0)$ is normal but not regular. In order to consider intersection problem, we need to construct a regular model first. Let $\pi : \mathcal{M} \to \mathcal{N}(x_0)$ be the blow up morphism along the unique closed $\mathbb{F}$-point of $\mathcal{N}(x_0)$. Then $\mathcal{M}$ is a 3-dimensional regular formal scheme. For a subset $H \subset \mathbb{B}$, denote by $\mathcal{Z}(H) = \pi^*\mathcal{Z}_{\mathcal{N}(x_0)}(H)$ (resp. $\mathcal{Y}(H) = \pi^*\mathcal{Y}_{\mathcal{N}(x_0)}(H)$) the direct pullback. When $H = \{x\}$ consists of only one element x, $\mathcal{Z}_{\mathcal{N}(x_0)}(x)$ and $\mathcal{Y}_{\mathcal{N}(x_0)}(x)$ are not Cartier divisors. However, we show that $\mathcal{Z}(x)$ and $\mathcal{Y}(x)$ are Cartier divisors on $\mathcal{M}$ (cf. Lemma 4.13.2 and Corollary 4.13.3).

Let $L \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 3, we now associate to L several integers: the arithmetic intersection number of $\mathcal{Z}$-cycles $\mathrm{Int}^{\mathcal{Z}}(L)$, of $\mathcal{Y}$-cycles $\mathrm{Int}^{\mathcal{Y}}(L)$, and the derived local density of L into some not self-dual lattices.

Let x_1, x_2, x_3 be a basis of L. Define the arithmetic intersection number of $\mathcal{Z}$-cycles to be

$$\mathrm{Int}^{\mathcal{Z}}(L) := \chi(\mathcal{M}, \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(x_2)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(x_3)}). \tag{3}$$

where $\mathcal{O}_{\mathcal{Z}(x_i)}$ denotes the structure sheaf of the special divisor $\mathcal{Z}(x_i)$, $\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}$ denotes the derived tensor product of coherent sheaves on $\mathcal{M}$, and χ denotes the Euler–Poincaré characteristic. The arithmetic intersection numbers of $\mathcal{Y}$-cycles $\mathrm{Int}^{\mathcal{Y}}(L)$ are defined by similar formula (3) but replacing $\mathcal{Z}$ by $\mathcal{Y}$. Both the numbers $\mathrm{Int}^{\mathcal{Z}}(L)$ and $\mathrm{Int}^{\mathcal{Y}}(L)$ are independent of the choice of the basis of L by the *linear invariance* property (cf. Corollary 5.6.3).

Now we define the local density and derived local density. For another quadratic $\mathbb{Z}_p$-lattice (of arbitrary rank) M, define $\mathrm{Rep}_{M,L}$ to be the scheme of integral representations, an $\mathbb{Z}_p$-scheme such that for any $\mathbb{Z}_p$-algebra R, $\mathrm{Rep}_{M,L}(R) = \mathrm{QHom}(L \otimes_{\mathbb{Z}_p} R, M \otimes_{\mathbb{Z}_p} R)$. Here QHom denotes the set of homomorphisms of quadratic modules. The local density associated to M and L is defined to be

$$\mathrm{Den}(M, L) = \lim_{d\to\infty} \frac{\#\mathrm{Rep}_{M,L}(\mathbb{Z}_p/p^d)}{p^{d\cdot \dim(\mathrm{Rep}(M,L))_{\mathbb{Q}_p}}}.$$

Let $H_0(p)$ and $H_0(p)^\vee$ be the following two quadratic lattices of rank 4 over $\mathbb{Z}_p$:

$$H_0(p) = \left\{ \begin{pmatrix} a & b \\ pc & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_p \right\}, \quad H_0(p)^\vee = \left\{ \begin{pmatrix} a & p^{-1}b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_p \right\}. \tag{4}$$

Here the quadratic forms of $H_0(p)$ and $H_0(p)^\vee$ are given by the determinant. Let $H_{2k}^+ = (H_2^+)^{\oplus k}$ be a self-dual lattice of rank $2k$, where the quadratic form on $H_2^+ = \mathbb{Z}_p^2$ is given by $(x, y) \in \mathbb{Z}_p^2 \mapsto xy$. Let H_1^+ be a self-dual lattice of rank 1 whose discriminant is a square in $\mathbb{Q}_p^\times$. There exist local density polynomials $\mathrm{Den}(X, H_0(p), L), \mathrm{Den}(X, H_0(p)^\vee, L) \in \mathbb{Q}[X]$ such that

$$\begin{aligned} \mathrm{Den}(X, H_0(p), L)\big|_{X=p^{-k}} &= \mathrm{Den}(H_0(p) \oplus H_{2k}^+, L), \\ \mathrm{Den}(X, H_0(p)^\vee, L)\big|_{X=p^{-k}} &= \mathrm{Den}(H_0(p)^\vee \oplus H_{2k}^+, L). \end{aligned} \tag{5}$$

Both polynomial vanishes at $X = 1$ since $L \subset \mathbb{B}$ is anisotropic and cannot be embedded into $H_0(p)$ and $H_0(p)^\vee$. Therefore we consider the (normalized) *derived local density*

$$\partial\mathrm{Den}\,(H_0(p), L) := -2 \cdot \frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \frac{\mathrm{Den}(X, H_0(p), L)}{\mathrm{Den}(H_0(p), H_2^+ \oplus H_1^+[p])},$$

$$\partial\mathrm{Den}\,(H_0(p)^\vee, L) := -2 \cdot \frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \frac{\mathrm{Den}(X, H_0(p)^\vee, L)}{\mathrm{Den}(H_0(p)^\vee, H_2^+ \oplus H_1^+[p^{-1}])},$$

Here for a quadratic lattice N and an element $a \in \mathbb{Q}_p$, the symbol $N[a]$ means another quadratic lattice with the same $\mathbb{Z}_p$-module as N but with the quadratic form multiplied by a.

Remark 1.2.1. It is pointed out by Chao Li that the extra scaling scalar 2 comes from the fact that the formal scheme $\mathcal{N}(x_0)$ is a double cover of the Rapoport–Zink space associated to the lattice $H_0(p)$.

Let $\mathcal{O}_{\mathbb{B}}$ be the maximal order of the division quaternion algebra $\mathbb{B}$. Our main theorem is the following two precise identities between several quantities just defined.

Theorem 1.2.2 (Theorem 5.6.7). *Let $L \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 3. Then*

$$\mathrm{Int}^{\mathcal{Z}}(L) = \partial\mathrm{Den}\left(H_0(p), L\right),$$

and

$$\mathrm{Int}^{\mathcal{Y}}(L) = \partial\mathrm{Den}\left(H_0(p)^\vee, L\right) - 1 = \partial\mathrm{Den}\left(H_0(p)^\vee, L\right) - \frac{p^7}{2(p+1)^2}\cdot \mathrm{Den}(\mathcal{O}_{\mathbb{B}}^\vee, L).$$

We remark that in an ongoing work, we have a precise conjecture for the general GSpin RZ space with maximal parahoric level which specialize to Theorem 1.2.2. We hope to establish more special cases to provide evidence for the general conjecture in the future. We also remark that the formulas in Theorem 1.2.2 is a manifestation of the duality phenomenon studied in [Cho22](cf. [Cho22, Conjecture 1.6]) in the GSpin group setting.

1.3. **Strategy of the proof.**

1.3.1. *Difference formula on the analytic side.* Let $\mathbf{L}$ be a vertex lattice, i.e., $p\mathbf{L}^\vee \subset \mathbf{L} \subset \mathbf{L}^\vee$. Let $\mathbf{M}$ be another quadratic lattice. Let $\langle x\rangle$ be a rank 1 quadratic lattice generated by x such that $n := \nu_p(q(x)) \geqslant 0$. In §3, we compute the difference of two local densities $\mathrm{Den}(\mathbf{L}, \mathbf{M} \obot \langle x\rangle)$ and $\mathrm{Den}(\mathbf{L}, \mathbf{M} \obot \langle p^{-1}x\rangle)$. The main result is Theorem 3.2.1. Specializing it to our situation, we get

Lemma 1.3.1 (Lemma 3.3.3). *Let $L^\flat \subset \mathbb{B}$ be an $\mathbb{Z}_p$-lattice of rank 2. Let $x \in \mathbb{B}$ be an element such that $\nu_p(q(x)) \geqslant 0$ and $x \perp L^\flat$. Then*

$$\begin{aligned}\partial\mathrm{Den}\left(H_0(p), L^\flat \obot \langle x\rangle\right) &- \partial\mathrm{Den}\left(H_0(p), L^\flat \obot \langle p^{-1}x\rangle\right)\\ &= \begin{cases} \partial\mathrm{Den}\left(\langle x\rangle[-1] \obot H_2^+[p], L^\flat\right), & \text{if } n = 0;\\[2ex] 2\cdot\partial\mathrm{Den}\left(\langle x\rangle[-1] \obot H_2^+[p], L^\flat\right) \\ \quad +\partial\mathrm{Den}\left(\langle x\rangle[-1] \obot H_2^+, L^\flat\right), & \text{if } n = 1;\\[2ex] 2\cdot\partial\mathrm{Den}\left(\langle x\rangle[-1] \obot H_2^+[p], L^\flat\right) \\ +2\cdot\partial\mathrm{Den}\left(\langle x\rangle[-1] \obot H_2^+, L^\flat\right), & \text{if } n \geqslant 2.\end{cases}\end{aligned} \tag{6}$$

The difference formula is related to the double coset decomposition of the lattice $H_0(p)$ under the left and right multiplication of $\Gamma_0(p) := H_0(p)^\times$. Let $x \in H_0(p)$ be a *primitive* element such that $\nu_p(\det(x)) = n \geqslant 0$, the primitivity of x means that $x \in H_0(p)\backslash pH_0(p)$. Then the double coset

$\Gamma_0(p)x\Gamma_0(p)$ has the following possibilities (Lemma 9.5.1):

$$\Gamma_0(p)\begin{pmatrix}1 & 0\\ 0 & p^n\end{pmatrix}\Gamma_0(p),\ \Gamma_0(p)\begin{pmatrix}p^n & 0\\ 0 & 1\end{pmatrix}\Gamma_0(p),\ \Gamma_0(p)\begin{pmatrix}0 & p^{n-1}\\ p & 0\end{pmatrix}\Gamma_0(p),\ \Gamma_0(p)\begin{pmatrix}0 & 1\\ p^n & 0\end{pmatrix}\Gamma_0(p).$$

When $n \geqslant 2$, the four cosets are different. When $n = 1$, the last two cosets are the same, therefore there are 3 different cosets. When $n = 0$, the last two cosets do not exist, while the first two are the same, therefore there is only 1 coset. That corresponds the number of terms appearing on the right hand side of the difference formula (6).

1.3.2. *Difference formula on the geometric side.* Denote by $\mathcal{M}_{\mathbb{F}}$ the base change $\mathcal{M} \times_W \mathbb{F}$. It can be viewed as the Cartier divisor of the ideal sheaf generated by p. There are four irreducible components of $\mathcal{M}_{\mathbb{F}}$ labeled by $\mathcal{M}^{\mathrm{FF}}, \mathcal{M}^{\mathrm{FV}}, \mathcal{M}^{\mathrm{VF}}$ and $\mathcal{M}^{\mathrm{VV}}$ (see §4.10). The superscripts here indicate the isogeny type (Frobenius or Verschiebung) of the first and second universal deformation of x_0 over the corresponding irreducible component. All four irreducible components are isomorphic to the formal completion of the scheme $\mathrm{Bl}_{(0,0)}\mathbb{A}^2_{\mathbb{F}}$ along its exceptional divisor. We have the following identity of Cartier divisors on $\mathcal{M}$ (Proposition 4.11.1):

$$\mathcal{M}_{\mathbb{F}} = 2\cdot \mathrm{Exc}_{\mathcal{M}} + \mathcal{M}^{\mathrm{FF}} + \mathcal{M}^{\mathrm{VV}} + \mathcal{M}^{\mathrm{FV}} + \mathcal{M}^{\mathrm{VF}}.$$

We prove the following lemma:

Lemma 1.3.2 (Lemma 6.4.1)**.** *Let $L^\flat \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 2. Let $x \in \mathbb{B}$ be another element such that $\nu_p(q(x)) \geqslant \max\{\mathrm{val}(L^\flat), 2\}$ and $x \perp L^\flat$, then*

$$\begin{aligned}\mathrm{Int}^{\mathcal{Z}}(L^\flat \oplus \langle x\rangle) - \mathrm{Int}^{\mathcal{Z}}(L^\flat \oplus \langle p^{-1}x\rangle) &= \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\mathcal{M}_{\mathbb{F}}})\\ &= \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\mathcal{M}^{\mathrm{FF}}}) + \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\mathcal{M}^{\mathrm{VV}}})\\ &\quad + \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\mathcal{M}^{\mathrm{FV}}}) + \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\mathcal{M}^{\mathrm{VF}}}).\end{aligned}$$

1.3.3. *Difference formula combined and the induction method.* For quadratic lattices of rank less or equal to 3, the local density polynomial has been computed explicitly in Yang's work [Yan98]. Combining with some calculations of intersection numbers in [GK93] which are blow-up invariant by §6.3, we prove the following:

Lemma 1.3.3 (Corollary 7.1.2)**.** *Let $L^\flat \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 2. Let $x \in \mathbb{B}$ be a nonzero element such that $x \perp L^\flat$ and $\nu_p(q(x)) \geqslant \max(L^\flat)$. Then*

$$\mathrm{Int}^{\mathcal{Z}}\left(L^\flat \oplus \langle x\rangle\right) - \mathrm{Int}^{\mathcal{Z}}\left(L^\flat \oplus \langle p^{-1}x\rangle\right) = \partial\mathrm{Den}\left(H_0(p), L^\flat \oplus \langle x\rangle\right) - \partial\mathrm{Den}\left(H_0(p), L^\flat \oplus \langle p^{-1}x\rangle\right).$$

Actually, there is a correspondence between the terms on the right hand side of the geometric difference formula (Lemma 1.3.2) and that of the analytic difference formula Lemma 1.3.1. Namely (Lemma 7.1.1),

$$\chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\mathcal{M}^{\mathrm{FF}}}) = \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\mathcal{M}^{\mathrm{VV}}}) = \partial\mathrm{Den}\left(\langle x\rangle[-1]\oplus H_2^+, L^\flat\right),$$
$$\chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\mathcal{M}^{\mathrm{FV}}}) = \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat)\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\mathcal{M}^{\mathrm{VF}}}) = \partial\mathrm{Den}\left(\langle x\rangle[-1]\oplus H_2^+[p], L^\flat\right).$$

Let (a_1, a_2, a_3) be the Gross–Keating invariant $\mathrm{GK}(L)$ of the quadratic lattice L where $a_1 \leqslant a_2 \leqslant a_3$. We prove Theorem 1.2.2 for $\mathcal{Z}$-cycles in the two base cases that the $\mathrm{GK}(L) = (0, 0, 1)$ and

$(0,1,1)$ by computing both sides explicitly (§3.5 and §7.2). Then the statement in Theorem 1.2.2 about $\mathcal{Z}$-cycles follows from Lemma 1.3.3 by inducting on the integer $n(L) = a_1 + a_2 + a_3$.

1.3.4. *Automorphism of $\mathcal{M}$.* There is a naturally defined automorphism $\iota^{\mathcal{M}}$ of the formal scheme $\mathcal{M}$ (§4.12). For a subset $H \subset \mathbb{B}$, we have $\mathcal{Y}(H) = \left(\iota^{\mathcal{M}}\right)^* (\mathcal{Z}(x_0 \cdot H))$. Combining the geometric difference formula for $\mathcal{Z}$-cycles and some identities between local densities (Lemma 3.4.1), we obtain:

Lemma 1.3.4 (Corollary 7.1.2)**.** *Let $L^\flat \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 2. Let $x \in \mathbb{B}$ be a nonzero element such that $x \perp L^\flat$ and $\nu_p(q(x)) \geqslant \max\{\max(L^\flat), 2\}$. Then*

$$\mathrm{Int}^{\mathcal{Y}}\left(L^\flat \oplus \langle x\rangle\right) - \mathrm{Int}^{\mathcal{Y}}\left(L^\flat \oplus \langle p^{-1}x\rangle\right) = \partial\mathrm{Den}\left(H_0(p)^\vee, L^\flat \oplus \langle x\rangle\right) - \partial\mathrm{Den}\left(H_0(p)^\vee, L^\flat \oplus \langle p^{-1}x\rangle\right).$$

Then the statement in Theorem 1.2.2 about $\mathcal{Y}$-cycles follows from Lemma 1.3.4 and similar induction method as the $\mathcal{Z}$-cycle case.

1.4. **Applications.**

1.4.1. *A conjecture of Kudla–Rapoport.* In 2006, Kudla and Rapoport defined CM cycles on the modular curve $X_0(N)$ for an arbitrary positive integer N. Roughly speaking, a CM cycle parameterizes a pair of CM isogenies (j, j') of two elliptic curves E, E' which admits a cyclic isogeny $\pi : E \to E'$ of degree N, where $j \in \mathrm{End}(E)$ and $j' \in \mathrm{End}(E')$ such that $j' \circ \pi = \pi \circ j$. It is conjectured that the local arithmetic intersection numbers of two CM cycles on the Rapoport–Zink space associated to the modular curve $X_0(N)$ are related to the derivatives of local densities in the flavor of (1). Our result can be applied to confirm this conjecture when N is odd and squarefree.

We only state the "local" theorem, from which the global one can be deduced easily. The key observation is the following: Suppose that the element $x_0 \in \mathbb{B}$ we picked satisfies $q(x_0) = N$. Then the strict transform $\widetilde{\mathcal{Z}}(1)$ of the special cycle $\mathcal{Z}(1)$ on $\mathcal{M}$ is isomorphic to the blow up of the Rapoport–Zink space associated to $X_0(N)$ along its maximal ideal. The pullback of special cycles $\mathcal{Z}(x)$ to $\widetilde{\mathcal{Z}}(1)$ coincides with CM cycles defined by Kudla–Rapoport. This fact can be viewed as a *geometric cancellation law* (cf. §8.2).

Let $M \subset \mathbb{B}^0$ (trace 0 elements in $\mathbb{B}$) be a rank 2 lattice with basis x, y. Define

$$\mathrm{Int}^{\mathrm{CM}}(M) = \chi(\mathcal{N}_0(N), \mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}(y)}).$$

This number is independent of the choice of basis x, y. Let $S \subset H_0(p)$ be the sublattice of trace 0 elements. Combining with *analytic cancellation law* (cf. §8.3) on the local densities, we obtain:

Theorem 1.4.1 (Theorem 8.5.2)**.** *Let $M \subset \mathbb{B}^0$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$\mathrm{Int}^{\mathrm{CM}}(M) = \partial\mathrm{Den}(S, M).$$

Remark 1.4.2. We remark here that Shi [Shi22] has a different method to calculate $\mathrm{Int}^{\mathrm{CM}}(M)$.

Remark 1.4.3. When N is squarefree, the modular curve $X_0(N)$ is a GSpin Shimura variety with almost self-dual level structure. In the language of He, Zhang and Zhu [HZZ25], there are two kinds of special cycles, $\mathcal{Z}$-cycles and $\mathcal{Y}$-cycles, on the associated Rapoport–Zink spaces (the definition on the integral models are similar). The CM cycles defined by Kudla and Rapoport for $X_0(N)$ correspond to the $\mathcal{Y}$-cycles, while the $\mathcal{Z}$-cycles on $X_0(N)$ are defined and studied by Shi, Sankaran and Yang [SSY23], and Zhu [Zhu25], [Zhu24a]. The relation between these two kinds of cycles can be found in [HZZ25, Proposition 5.5.1]. If we replace $X_0(N)$ by a Shimura curve Sh_B associated to a quaternion algebra B, which we regard as a GSpin Shimura variety with almost self-dual level structure again, then the intersection of $\mathcal{Y}$-cycles was considered in [KR00, KRY06].

1.4.2. *Arithmetic intersections of Hecke correspondences.* Let $\mathbb{V}$ be a rank 4 *incoherent* quadratic space over $\mathbb{A}$ given by the following:

$$\mathbb{V}_v = V_v = \mathrm{M}_2(\mathbb{Q}_v) \text{ if } v < \infty, \text{ and } \mathbb{V}_\infty \text{ is positive definite.} \tag{7}$$

Given a Schwartz function $\Phi = \Phi_\infty \otimes \Phi_f \in \mathscr{S}(\mathbb{V}^3)$ where Φ_∞ is given by the standard Gaussian function on $\mathbb{V}_\infty^3$. There is a classical incoherent Eisenstein series $E(g, s, \Phi)$ (§9.9) on the group GSp_6. Denote by $E_T(g, s, \Phi)$ the T-th Fourier expansion of it. The central value of $E(g, s, \Phi)$ is 0 by the incoherence. Let $E_T'(g, 0, \Phi)$ be the derivative of $E_T(g, s, \Phi)$ at 0.

Let N be an odd squarefree positive integer. Let $\mathcal{M}_0(N)$ be the blow up of the Deligne–Mumford stack $\mathcal{X}_0(N) \times_{\mathbb{Z}} \mathcal{X}_0(N)$ along its supersingular points with residue field characteristic $p|N$. For a positive integer m, we define a Cartier divisor $\widehat{\mathsf{T}}(m)$ (cf. (104)) on $\mathcal{M}_0(N)$ whose generic fiber equals to the classically defined m-th Hecke correspondence on $X_0(N)^2$ (Lemma 9.8.2).

Theorem 1.4.4 (Theorem 9.10.1). *Let m_1, m_2, m_3 be three positive integers such that there is no positive definite binary quadratic form over $\mathbb{Z}$ which represents the three integers m_1, m_2, m_3. Then*

$$\left(\widehat{\mathsf{T}}(m_1) \cdot \widehat{\mathsf{T}}(m_2) \cdot \widehat{\mathsf{T}}(m_3)\right) = \sum_{T} -2E_T'\left(1, 0, \Phi_\infty \otimes \mathbf{1}_{\left(H_0(N)\otimes\hat{\mathbb{Z}}\right)^3}\right),$$

where the summation ranges over all the half-integral symmetric positive definite 3×3 matrices T with diagonal elements m_1, m_2, m_3.

Remark 1.4.5. For each single non-degenerate positive definite matrix $T \in \mathrm{Sym}_3(\mathbb{Q})$, we also prove a (semi)-global Arithmetic Siegel–Weil formula in §9.12 following the idea in [LZ22b, §12].

The condition on m_1, m_2, m_3 guarantees that there is no self-intersections between the three divisors $\widehat{\mathsf{T}}(m_i)$ on the generic fiber. In the works of Yuan–Zhang–Zhang [YZZ23], they use "regular" test functions (cf. Definition 4.4.1 of *loc. cit.*) to avoid the self-intersection. Using this method, they proved the (semi-global) arithmetic Siegel–Weil formula on the product of modular curves at a place p where the modular curve has good reduction (cf. Theorem 5.4.3 of *loc. cit.*). Our work gives a generalization of their formula at a prime p where the level structure is given by $\Gamma_0(p)$.

Let p be an odd prime. Let $U = \Gamma_0(p) \times U^p$ be an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$. Let $K = U \times_{\mathbb{G}_m} U$. We define a regular integral model $\mathcal{M}_{K,(p)}$ over $\mathbb{Z}_{(p)}$ for the product of modular curves $Y_U \times Y_U$. Using Weil representation, we can vary the Hecke correspondence and define a divisor $\widehat{\mathsf{T}}(g, \phi)$ on $\mathcal{M}_{K,(p)}$ where $g \in \mathrm{GL}_2(\mathbb{A})$ and $\phi \in \mathscr{S}(\mathbb{V})$ following the works of Kudla [Kud04] and Yuan–Zhang–Zhang.

Theorem 1.4.6 (Theorem 9.11.2). *Let $\phi_i \in \mathscr{S}(\mathbb{V})$ be three Schwartz functions satisfying*

(1) *$\phi_{i,\infty}$ is the standard Gaussian function on $\mathbb{V}_\infty$.*
(2) *$\phi_{i,p} = 1_{H_0(p)}$ or $1_{H_0(p)^\vee}$ and ϕ_i^p is invariant under the group K^p.*
(3) *There exists a finite place v prime to p such that the Schwartz function $\phi_v = \phi_{1,v} \otimes \phi_{2,v} \otimes \phi_{3,v} \in \mathscr{S}(\mathbb{V}_v^3)$ is regularly supported in the sense of* [YZZ23, Definition 4.4.1].

Let $\Phi = \phi_1 \otimes \phi_2 \otimes \phi_3 \in \mathscr{S}(\mathbb{V}^3)$. *Then for all elements* $g = (g_1, g_2, g_3) \in (\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2)(\mathbb{A})$ *such that* $g_v = \mathbf{1}_v$ *and* $g_p = \mathbf{1}_p$. *We have*

$$\left(\widehat{\mathsf{T}}(g_1, \phi_1) \cdot \widehat{\mathsf{T}}(g_2, \phi_2) \cdot \widehat{\mathsf{T}}(g_3, \phi_3)\right)_p = \sum_{T:\mathrm{Diff}(T)=\{p\}} -2E'_T(g, 0, \Phi), \quad \text{if } \Phi_p = 1_{H_0(p)^3} \text{ or } 1_{H_0(p)^{\vee 3}}. \tag{8}$$

The set $\mathrm{Diff}(T)$ *consists of all finite places* l *where* T *is not represented by* $\mathbb{V}_l$.

Remark 1.4.7. During the proof of the above theorem, we find that the case $\Phi_p = 1_{H_0(p)^3}$ corresponds to the global intersections of $\mathcal{Z}$-cycles on $\mathcal{M}_{K,(p)}$, while the case $\Phi_p = 1_{H_0(p)^{\vee 3}}$ corresponds to the global intersections of $\mathcal{Y}$-cycles.

1.5. **The structure of the paper.** In Part 1, we introduce notations on quadratic lattices and local density (§2). Then we establish a difference formula of local density and apply it to cases that we are interested (§3).

In Part 2, we define the Rapoport–Zink space and special cycles on it in §4. Then we study the intersection of special divisors and the exceptional divisor in §5. The main results in this section allows us to give a complete decomposition of the special divisors. We also prove the linear invariance of the derived special cycles. In §6, we establish the difference formula on the geometric side. In §7, we combine the geometric and analytic difference formulas and also calculate the base cases to prove the main theorem (Theorem 1.2.2).

Finally in Part 3, we give two applications of our results: A conjecture of Kudla and Rapoport (§8) and the arithmetic intersections of Hecke correspondences (§9).

1.6. **Acknowledgments.** We would like to thank Chao Li, Yifeng Liu, Michael Rapoport, Wei Zhang and Zhiyu Zhang for helpful conversations and comments. We also would like to thank the anonymous referees for the valuable comments that greatly improve the paper.

1.7. **Notations.** Throughout this article, we fix an odd prime p (see Remarks 3.4.3 and 5.4.5 for why we require $p > 2$). Let $F/\mathbb{Q}_p$ be a finite extension, with ring of integers $\mathcal{O}_F$, and uniformizer π. Let $\nu_\pi : F \to \mathbb{Z} \cup \{\infty\}$ be the valuation map on F which maps π to 1. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. Let $\breve{F}$ be the completion of the maximal unramified extension of F. Let $\mathcal{O}_{\breve{F}}$ be the integer ring of the field $\breve{F}$.

1.7.1. *On crystalline sites.* For a scheme or formal scheme S over $\mathcal{O}_{\breve{F}}$, denote by $\mathrm{NCRIS}_{\mathcal{O}_{\breve{F}}}(S/\mathrm{Spec}\,\mathcal{O}_{\breve{F}})$ the big fppf nilpotent crystalline site of S over $\mathcal{O}_{\breve{F}}$ (cf. [FGL08, Definition B.5.7.]), the definition is the same as the crystalline site defined in the works of Berthelot, Breen and Messing [BBM82, §1.1.1.] except that we replace the pd-structure by $\mathcal{O}_F$-pd-structure [FGL08, Definition B.5.1.] (Notice that they are equivalent when $F = \mathbb{Q}_p$). Denote by $\mathcal{O}_S^{\mathrm{crys}}$ the structure sheaf in this site. For a point $z \in S(\mathbb{F})$, let $\widehat{\mathcal{O}}_{S,z}$ be the complete local ring of S at the point z. Let $\mathrm{Nilp}_{\mathcal{O}_{\breve{F}}}$ be the category of $\mathcal{O}_{\breve{F}}$-schemes on which π is locally nilpotent. For an object S in $\mathrm{Nilp}_{\mathcal{O}_{\breve{F}}}$, we use $\overline{S}$ to denote the scheme $S \times_{\mathcal{O}_{\breve{F}}} \mathbb{F}$.

1.7.2. *On Grothendieck* K*-groups.* For a noetherian formal scheme X together with a closed formal subscheme Y, denote by $K_0^Y(X)$ the Grothendieck group of finite complexes of coherent locally free $\mathcal{O}_X$-modules acyclic outside Y . We use $K_0(X)$ to denote $K_0^X(X)$. Let $K_0'(Y)$ be the Grothendieck group of coherent sheaves of $\mathcal{O}_Y$-modules on Y.

Throughout this article, we adopt the notational convention of writing $\mathcal{L}$, for a line bundle on a (formal) scheme X, to also represent the class $[\mathcal{O}_X] - [\mathcal{L}]$ in the Grothendieck group $K_0(X)$.

Let $(X, \mathcal{O}_X)$ be a noetherian formal scheme over $\mathrm{Spf}\,\mathcal{O}_{\breve{F}}$, define its dimension to be $\sup\limits_{x\in X(\mathbb{F})} \dim \mathcal{O}_{X,x}$. For a closed formal subscheme $Y \hookrightarrow X$, define the codimension of Y in X to be $\mathrm{codim}_X Y = \inf\limits_{x\in Y(\mathbb{F})} \{\dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,x}\}$. Notice that both the notions of dimension and codimension agree with that for schemes. Denote by $\mathrm{F}^i K_0^Y(X)$ the codimension i descending filtration on $K_0^Y(X)$

$$\mathrm{F}^i K^Y(X) = \bigcup_{\substack{Z\subset Y \\ \mathrm{codim}_X Z \geqslant i}} \mathrm{Im}\left(K_0^Z(X) \to K_0^Y(X)\right). \tag{9}$$

Denote by $\mathrm{Gr}^i K_0^Y(X)$ its i-th graded piece. For an object $A \in \mathrm{F}^i K_0^Y(X)$, denote by $[A]$ its image in $\mathrm{Gr}^i K_0^Y(X)$. There is also an ascending filtration $\mathrm{F}_i K_0'(X)$ on $K_0'(X)$

$$\mathrm{F}_i K_0'(X) = \bigcup_{\substack{Z\subset X \\ \dim Z \leqslant i}} \mathrm{Im}\left(K_0'(Z) \to K_0'(X)\right). \tag{10}$$

We also have a cup product $\cdot$ on $K_0^Y(X)_{\mathbb{Q}} := K_0^Y(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ defined by tensor product of complexes. Under the identification $K_0^Y(X) \xrightarrow{\sim} K_0'(Y)$, the cup product is derived tensor product.

Part 1. Analytic side

2. Quadratic lattices and local densities

2.1. **Notations on quadratic spaces and lattices.** A quadratic space (U, q_U) over F is a finite dimensional vector space U over F equipped with a quadratic form $q_U : U \to F$, the quadratic form q_U induces a symmetric bilinear form given by

$$\begin{aligned} (\cdot,\cdot) : U \times U &\to F, \\ (x, y) &\mapsto q_U(x + y) - q_U(x) - q_U(y). \end{aligned} \tag{11}$$

We say a quadratic space U is non-degenerate if the bilinear form $(\cdot,\cdot)$ is non-degenerate.

An isometry between two quadratic spaces (U, q_U) and $(U', q_{U'})$ is a linear isomorphism $\phi : U \to U'$ preserving quadratic forms, i.e., $q_{U'}(\phi(x)) = q_U(x)$ for any $x \in U$. In that case, we say U and U' are called isometric.

Given by a quadratic space (U, q_U) over F and a nonzero number $s \in F$, denote by $(U[s], q_U[s])$ the quadratic space whose underlying linear space $U[s] = U$, and the quadratic form $q_U[s]$ defined as $q_U[s](x) = s \cdot q_U(x)$, for all $x \in U$.

A quadratic lattice (L, q_L) is a finite free $\mathcal{O}_F$-module equipped with a quadratic form $q_L : L \to F$. The quadratic form q_L also induces a symmetric bilinear form $L \times L \xrightarrow{(\cdot,\cdot)} F$ by similar formula (11). The quadratic form q_L can be extended to the linear space $L_F := L \otimes_{\mathcal{O}_F} F$. We say a quadratic lattice L is non-degenerate if the quadratic space L_F is non-degenerate. In the following paragraphs, we will always assume the lattice is non-degenerate.

For a nonzero number $s \in F$, denote by $(L[s], q_L[s])$ the quadratic lattice whose underlying lattice $L[s] = L$, and the quadratic form $q_L[s]$ defined as $q_L[s](x) = s \cdot q_L(x)$, for all $x \in L$.

For a quadratic lattice L and a sublattice $M \subset L$, define $M^{\perp} := \{x \in L : (x, M) = 0\}$. For two quadratic lattices (L_1, q_1) and (L_2, q_2), define $L_1 \oplus L_2$ to be the quadratic lattice whose underlying $\mathcal{O}_F$-lattice is $L_1 \oplus L_2$, and the quadratic form $q = q_1 \oplus q_2$.

We say a quadratic lattice is integral if $q_L(x) \in \mathcal{O}_F$ for all $x \in L$. For an integral lattice L, define $L^\vee = \{x \in L \otimes_{\mathcal{O}_F} F : (x, L) \subset \mathcal{O}_F\}$. Define its fundamental invariants to be the unique sequence of integers $(a_1, \cdots, a_n)$ such that $0 \leqslant a_1 \leqslant \cdots \leqslant a_n$, and $L^\vee/L \simeq \oplus_{i=1}^n \mathcal{O}_F/\pi^{a_i}$ as $\mathcal{O}_F$-modules. Define

$$\min(L) = a_1, \quad \max(L) = a_n.$$

Definition 2.1.1. We say a quadratic lattice L is a vertex lattice if it is integral and all the fundamental invariants a_i satisfy that $0 \leqslant a_i \leqslant 1$, i.e.,

$$\pi L^\vee \subset L \subset L^\vee.$$

When all the fundamental invariants are 0, i.e., $L = L^\vee$, we say the quadratic lattice L is self-dual.

Let $a_1' < \cdots < a_r'$ be all the different numbers appearing in $a_1, \cdots, a_n$. There exists a Jordan decomposition of the quadratic lattice L as follows,

$$L \simeq \oplus_{i=1}^r L_i,$$

where $L_i[\pi^{-a_i'}]$ is a self-dual lattice.

Lemma 2.1.2. *The Jordan decomposition is unique in the sense that if we have two decompositions $L \simeq \oplus_{i=1}^r L_i$ and $L \simeq \oplus_{i=1}^{r'} L_i'$, where $L_i[\pi^{-a_i'}], L_i'[\pi^{-b_i'}]$ are self-dual quadratic lattices for some integers $a_i', b_i' \geqslant 0$ and $a_1' < \cdots < a_r', b_1' < \cdots < b_r'$. Then we must have $r' = r$ and $L_i' \simeq L_i$ (hence $a_i' = b_i'$) for all i.*

Proof. This is proved in the works of R. Schulze-Pillot [SP21, Corollary 5.21]. ☐

Example 2.1.3. Let (U, q_U) be a quadratic space. For an element $x \in U$, we use $\langle x \rangle$ to denote the rank 1 quadratic lattice generated by x. It is non-degenerate if $q(x) \neq 0$. It is integral if $q(x) \in \mathcal{O}_F$. Its fundamental invariant is $\nu_\pi(q(x))$. When p is odd, the lattice $\langle x \rangle$ is self-dual if and only if $\nu_\pi(q(x)) = 0$.

2.2. **Some invariants.** Let (U, q_U) be a quadratic space. Let's assume that $\dim_F U = n$ and the symmetric bilinear form $(\cdot, \cdot)$ is nondegenerate. Let $\{x_i\}_{i=1}^n$ be a basis of U, and $t_{ij} = \frac{1}{2}(x_i, x_j)$, we define the discriminant of the quadratic space U to be:

$$\mathrm{disc}(U) = (-1)^{n(n-1)/2} \det((t_{ij})) \in F^\times/(F^\times)^2.$$

If $\{x_i\}_{i=1}^n$ is an orthogonal basis of U then $t_{ij} = 0$ if $i \neq j$ and $t_{ii} \neq 0$ by the nondegeneracy of $(\cdot, \cdot)$. The Hasse invariant of the quadratic space U is

$$\epsilon(U) = \prod_{i<j} (t_{ii}, t_{jj})_F,$$

For a quadratic lattice L, we use $\mathrm{disc}(L)$ and $\epsilon(L)$ to denote the corresponding invariants on the quadratic space $L_F = L \otimes_{\mathcal{O}_F} F$. Recall that when p is odd, quadratic spaces U over F are classified by the following three invariants:

$$\dim_F U, \quad \mathrm{disc}(U), \quad \epsilon(U).$$

i.e., two quadratic spaces U and U' are isometric if and only if the above three invariants for U and U' are the same.

Let $\chi = \left(\frac{\cdot}{\pi}\right) : F^\times/(F^\times)^2 \to \{\pm 1, 0\}$ be the quadratic residue symbol. For a quadratic space U, define $\chi(U) = \chi(\operatorname{disc} U)$. For a quadratic lattice L, define $\chi(L) = \chi(\operatorname{disc} L_F)$.

When p is odd, the quadratic space U admits a self-dual sub-lattice if and only if $\epsilon(U) = +1$ and $\chi(U) \neq 0$, we will use H_k^ε to denote the unique self-dual lattice of rank k and

$$\chi(H_k^\varepsilon) := \chi(\operatorname{disc}(H_k^\varepsilon)) = \varepsilon.$$

2.3. **Local densities of quadratic lattices.**

Definition 2.3.1. Let L, M be two integral quadratic $\mathcal{O}_F$-lattices. Let $\mathrm{Rep}_{M,L}$ be the scheme of integral representations, an $\mathcal{O}_F$-scheme such that for any $\mathcal{O}_F$-algebra R,

$$\mathrm{Rep}(M, L)(R) = \mathrm{QHom}(L \otimes_{\mathcal{O}_F} R, M \otimes_{\mathcal{O}_F} R),$$

where QHom denotes the set of quadratic module homomorphism.

Definition 2.3.2. Let L, M be two quadratic $\mathcal{O}_F$-lattices (not necessarily integral) with rank n, m respectively. Let $a \geqslant 0$ be an integer such that $L[\pi^a]$ and $M[\pi^a]$ are both integral. The local density of integral representations is defined to be

$$\mathrm{Den}(M, L) = q^{-a \cdot n(n+1)/2} \lim_{d \to \infty} \frac{\#\mathrm{Rep}(M[\pi^a], L[\pi^a])(\mathcal{O}_F/\pi^d)}{q^{d \cdot \dim(\mathrm{Rep}(M,L))_F}}.$$

This number is independent of the choice of a by [Kit99, Proposition 5.6.1].

Remark 2.3.3. If L, M have rank n, m respectively and the generic fiber $(\mathrm{Rep}_{M,L})_F \neq \varnothing$, then $n \leqslant m$ and

$$\dim(\mathrm{Rep}(M, L))_F = \dim \mathrm{O}_m - \dim \mathrm{O}_{m-n} = \binom{m}{2} - \binom{m-n}{2} = mn - \frac{n(n+1)}{2}.$$

It is well-known that there exists a local density polynomial $\mathrm{Den}(X, M, L) \in \mathbb{Q}[X]$ such that

$$\mathrm{Den}(q^{-k}, M, L) = \mathrm{Den}(M \oplus H_{2k}^+, L)$$

for all non-negative integers k. For our case, one can see this via Lemma 3.3.1 and an induction argument.

Definition 2.3.4. Let L, M be two quadratic $\mathcal{O}_F$-lattices. Let $\mathrm{PRep}_{M,L}$ be the $\mathcal{O}_F$-scheme of primitive integral representations such that for any $\mathcal{O}_F$-algebra R,

$$\mathrm{PRep}(M, L)(R) = \{\phi \in \mathrm{Rep}(M, L)(R) : \phi \text{ is an isomorphism between } L_R \text{ and a direct summand of } M_R\}.$$

where L_R (resp. M_R) is $L \otimes_{\mathcal{O}_F} R$ (resp. $M \otimes_{\mathcal{O}_F} R$). The primitive local density is defined to be

$$\mathrm{Pden}(M, L) = \lim_{d \to \infty} \frac{\#\mathrm{PRep}(M, L)(\mathcal{O}_F/\pi^d)}{q^{d \cdot \dim(\mathrm{Rep}_{M,L})_F}}.$$

Remark 2.3.5. For a positive integer d, a homomorphism $\phi \in \mathrm{Rep}(M, L)(\mathcal{O}_F/\pi^d)$ or $\mathrm{Rep}(M, L)(\mathcal{O}_F)$ is primitive if and only if $\overline{\phi} := \phi \bmod \pi \in \mathrm{PRep}(M, L)(\mathcal{O}_F/\pi)$, which is equivalent to $\dim_{\mathbb{F}_q}(\phi(L) + \pi \cdot M)/\pi \cdot M = \mathrm{rank}_{\mathcal{O}_F}(L)$. This agrees with the definition of primitive local density in [CY20, §3.1]

Example 2.3.6. Let k be a positive integer. Let $\varepsilon \in \{\pm\}$. Let $s \in \mathcal{O}_F$ be a nonzero number. Let $\langle x \rangle$ be a rank 1 quadratic lattice such that $q(x) = s$. It has been calculated explicitly that ([LZ22b, (3.3.2.1)])

$$\mathrm{Pden}(H_k^\varepsilon, \langle x \rangle) = \begin{cases} 1 - q^{1-k}, & \text{when } k \text{ is odd and } \pi \mid s; \\ 1 + \varepsilon\chi(s)q^{(1-k)/2}, & \text{when } k \text{ is odd and } \pi \nmid s; \\ (1 - \varepsilon q^{-k/2})(1 + \varepsilon q^{1-k/2}), & \text{when } k \text{ is even and } \pi \mid s; \\ 1 - \varepsilon q^{-k/2}, & \text{when } k \text{ is even and } \pi \nmid s. \end{cases} \tag{12}$$

3. Difference formula on the analytic side

3.1. **Primitive elements in vertex lattices.** Let (L, q) be a vertex quadratic lattice such that $L^\vee \neq L$ or $\pi^{-1}L$. Let $(\cdot, \cdot)$ be the bilinear form on L. There exists two self-dual lattices (H_1, q_1) and (H_2, q_2) such that $\mathrm{rank}_{\mathcal{O}_F} H_1, \mathrm{rank}_{\mathcal{O}_F} H_2 \geqslant 1$.

$$L \simeq H_1[\pi] \obot H_2. \tag{13}$$

By the uniqueness of the Jordan decomposition, the lattices H_1 and H_2 are unique under isometric equivalence. We will fix an isomorphism (13) in the following paragraphs.

We say an element $x \in L$ is primitive in L if $x \notin \pi L$. Let $\mathrm{O}(L)$ be the orthogonal group of the lattice L, i.e., $\mathrm{O}(L) = \{g \in \mathrm{Aut}_{\mathcal{O}_F}(L) : (g(x_1), g(x_2)) = (x_1, x_2) \text{ for all } x_1, x_2 \in L\}$.

Lemma 3.1.1. *Let $x = x_1 + x_2 \in L$ be a primitive element where $x_1 \in H_1[\pi]$ and $x_2 \in H_2$.*

(a) *If x_2 is a primitive element in H_2, there exists an element $g \in \mathrm{O}(L)$ such that $g(x) \in H_2$.*
(b) *If x_2 is not a primitive element in H_2, there exists an element $g \in \mathrm{O}(L)$ such that $g(x) \in H_1[\pi]$.*

Proof. We first prove (a). If $\mathrm{rank}_{\mathcal{O}_F} H_2 = 1$, the primitivity of x_2 implies that $\nu_\pi(q(x)) = 0$. We have the following decomposition

$$L = \langle x \rangle \obot \langle x \rangle^\perp.$$

By the uniqueness of the Jordan decomposition (Lemma 2.1.2), there exist two isometric maps $\phi_1 : \langle x \rangle \xrightarrow{\sim} H_2$ and $\phi_2 : \langle x \rangle^\perp \xrightarrow{\sim} H_1[\pi]$. Define $g \in \mathrm{O}(L)$ to be the composition $L = \langle x \rangle^\perp \obot \langle x \rangle \xrightarrow{\phi_1 \obot \phi_2} H_1[\pi] \obot H_2 \xrightarrow{(13)} L$, then $g(x) \in H_2$.

If $\mathrm{rank}_{\mathcal{O}_F} H_2 \geqslant 2$. The primitivity of x_2 implies that there exists an element $y \in H_2$ such that $\mathbb{F}_q \cdot \overline{x} + \mathbb{F}_q \cdot \overline{y} \subset L/\pi L$ is a non-degenerate quadratic subspace since $\overline{x_2} \neq 0$ and H_2 is self-dual. Then the lattice $M := \mathcal{O}_F \cdot x + \mathcal{O}_F \cdot y$ is self-dual. We have the following decomposition

$$L = M \obot M^\perp.$$

Then $M^\perp$ is a vertex lattice, therefore there exists two self-dual lattices M_1 and M_2 such that $M^\perp \simeq M_1[\pi] \obot M_2$. We get $L \simeq M_1[\pi] \obot M \obot M_2$. Lemma 2.1.2 implies that there exist two isometric maps $\phi_1 : M_1 \xrightarrow{\sim} H_1$ and $\phi_2 : M \obot M_2 \xrightarrow{\sim} H_2$. Define $g \in \mathrm{O}(L)$ to be the composition $L \simeq M_1[\pi] \obot M \obot M_2 \xrightarrow{\phi_1 \obot \phi_2} H_1[\pi] \obot H_2 \xrightarrow{(13)} L$, then $g(x) \in H_2$.

Now we prove (b). We know that x_1 is primitive in H_1 since x_2 is not primitive in H_2. If $\mathrm{rank}_{\mathcal{O}_F} H_1 = 1$, the primitivity of x_1 implies that $\nu_\pi(q_1[\pi](x)) = 1$. Notice that for all elements

$l \in L$, we have $\nu_\pi((x,l)) = \nu_\pi((x_1,l)+(x_2,l)) \geqslant \min\{\nu_\pi((x_1,l)), \nu_\pi((x_2,l))\} \geqslant 1$. We have the following decomposition

$$L = \langle x \rangle \obot \langle x \rangle^\perp.$$

By the uniqueness of the Jordan decomposition (Lemma 2.1.2), there exist two isometric maps $\phi_1 : \langle x \rangle \xrightarrow{\sim} H_1[\pi]$ and $\phi_2 : \langle x \rangle^\perp \xrightarrow{\sim} H_2$. Define $g \in \mathrm{O}(L)$ to be the composition $L = \langle x \rangle \obot \langle x \rangle^\perp \xrightarrow{\phi_1 \obot \phi_2} H_1[\pi] \obot H_2 \xrightarrow{(13)} L$, then $g(x) \in H_1[\pi]$.

If $\mathrm{rank}_{\mathcal{O}_F} H_2 \geqslant 2$. The primitivity of x_1 in H_1 implies that there exists an element $y \in H_1$ such that $\mathbb{F}_q \cdot \overline{x_1} + \mathbb{F}_q \cdot \overline{y} \subset H_1/\pi H_1$ is a non-degenerate quadratic subspace since $\overline{x_1} \neq 0$ and H_1 is self-dual. Let $M := \mathcal{O}_F \cdot x + \mathcal{O}_F \cdot y \subset L$. The inner product matrix of M under the basis $\{x, y\}$ is

$$\begin{pmatrix} \pi q_1(x_1) + q_2(x_2) & \frac{\pi}{2}(x_1,y) \\ \frac{\pi}{2}(x_1,y) & \pi q_1(y) \end{pmatrix} = \pi \cdot \begin{pmatrix} q_1(x_1) & \frac{1}{2}(x_1,y) \\ \frac{1}{2}(x_1,y) & q_1(y) \end{pmatrix} + \pi^2 \begin{pmatrix} q_2(x_2/\pi) & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore $M^\vee/M \simeq \mathcal{O}_F/\pi \oplus \mathcal{O}_F/\pi$. Notice that for all elements $l \in L$ and $m = ax + by \in M$, we have $\nu_\pi((m,l)) = \nu_\pi(a(x,l) + b(y,l)) \geqslant \min\{\nu_\pi((x_1,l)), \nu_\pi((x_2,l)), \nu_\pi((y,l))\} \geqslant 1$. Then we have a decomposition

$$L = M \obot M^\perp.$$

Then $M^\perp$ is a vertex lattice, therefore there exists two self-dual lattices M_1 and M_2 such that $M^\perp \simeq M_1[\pi] \obot M_2$. We get $L \simeq M \obot M_1[\pi] \obot M_2$. Lemma 2.1.2 implies that there exist two isometric maps $\phi_1 : M \obot M_1[\pi] \xrightarrow{\sim} H_1[\pi]$ and $\phi_2 : M_2 \xrightarrow{\sim} H_2$. Define $g \in \mathrm{O}(L)$ to be the composition $L \simeq M \obot M_1[\pi] \obot M_2 \xrightarrow{\phi_1 \obot \phi_2} H_1[\pi] \obot H_2 \xrightarrow{(13)} L$, then $g(x) \in H_2$. □

Lemma 3.1.2. *Let $n \geqslant 2$ be a positive integer and $\varepsilon \in \{\pm\}$.*

(a) *For a primitive element $x \in H_n^\varepsilon$, we have*

$$\langle x \rangle^\perp \simeq \begin{cases} \langle x \rangle[-1] \obot H_{n-2}^\varepsilon, & \text{if } n \geqslant 3; \\ \langle x \rangle[-\varepsilon], & \text{if } n = 2. \end{cases}$$

(b) *For a primitive element $x \in H_n^\varepsilon[\pi]$, we have*

$$\langle x \rangle^\perp \simeq \begin{cases} \langle x \rangle[-1] \obot H_{n-2}^\varepsilon[\pi], & \text{if } n \geqslant 3; \\ \langle x \rangle[-\varepsilon], & \text{if } n = 2. \end{cases}$$

Proof. The case $n = 2$ can be verified by hand. For $n \geqslant 3$, the quadratic lattice H_n^ε is isometric to the quadratic lattice $H_2^+ \obot H_{n-2}^\varepsilon$. For (a), we can assume $x \in H_2^+$, then it's easy to see that the orthogonal complement of x in H_2^+ is isometric to $\langle x \rangle[-1]$. The statement (b) follows from (a). □

3.2. **Difference formula for vertex lattices.** The main goal of this section is to prove the following formula.

Theorem 3.2.1. *Let $n_1, n_2 \geqslant 2$ be two integers. Let $\varepsilon_1, \varepsilon_2 \in \{\pm\}$. Let $L = H_{n_1}^{\varepsilon_1}[\pi] \obot H_{n_2}^{\varepsilon_2}$. Let M be a quadratic lattice of rank m. Let $\langle x \rangle$ be a rank 1 quadratic lattice generated by x such that*

$n := \nu_\pi(q(x)) \geqslant 0$. *Let*

$$L_1 := \begin{cases} H_{n_1-2}^{\varepsilon_1}[\pi] \oplus H_{n_2}^{\varepsilon_2} \oplus \langle x\rangle[-1], & \text{if } n_1 \geqslant 3; \\ H_{n_2}^{\varepsilon_2} \oplus \langle x\rangle[-\varepsilon_1], & \text{if } n_1 = 2. \end{cases}$$

$$L_2 := \begin{cases} H_{n_1}^{\varepsilon_1}[\pi] \oplus H_{n_2-2}^{\varepsilon_2} \oplus \langle x\rangle[-1], & \text{if } n_2 \geqslant 3; \\ H_{n_1}^{\varepsilon_1}[\pi] \oplus \langle x\rangle[-\varepsilon_2], & \text{if } n_2 = 2. \end{cases}$$

Then we have

$$\begin{aligned} &\mathrm{Den}\,(L, M \oplus \langle x\rangle) - q^{m+2-n_1-n_2} \cdot \mathrm{Den}\left(L, M \oplus \langle \pi^{-1}x\rangle\right) \\ &= q^{m+1-n_2} \cdot \mathrm{Den}\,(L_1, M) \cdot \mathrm{Pden}\left(H_{n_1}^{\varepsilon_1}, \langle x\rangle[\pi^{-1}]\right) + \mathrm{Den}\,(L_2, M) \cdot \mathrm{Pden}\left(H_{n_2}^{\varepsilon_2}, \langle x\rangle\right). \end{aligned} \tag{14}$$

Remark 3.2.2. In the formula (14), some non-integral quadratic lattices might appear, such as $\langle \pi^{-1}x\rangle$ and $\langle x\rangle[\pi^{-1}]$. We take the corresponding local density term to be 0 when such non-integral quadratic lattice appear by Definition 2.3.2.

3.2.1. *Decomposition of some sets.*

Lemma 3.2.3. *Let $d \geqslant n+1$ be an integer. Then we have a decomposition of the set* $\mathrm{Rep}(L, \langle x\rangle)(\mathcal{O}_F/\pi^d)$ *as follows,*

$$\begin{aligned} \mathrm{Rep}(L, \langle x\rangle)(\mathcal{O}_F/\pi^d) &= \bigsqcup_{i=0}^{[n/2]} \bigsqcup_{t \in \mathcal{O}_F/(\pi^i)} \mathrm{PRep}\left(L, \langle \pi^{-i}x\rangle[1 + \pi^d q(x)^{-1}t]\right)(\mathcal{O}_F/\pi^{d-i}) \\ &\simeq \bigsqcup_{i=0}^{[n/2]} \bigsqcup_{t \in \mathcal{O}_F/(\pi^i)} \mathrm{PRep}\left(L, \langle \pi^{-i}x\rangle\right)(\mathcal{O}_F/\pi^{d-i}). \end{aligned}$$

Proof. Let $\phi \in \mathrm{Rep}(L, \langle x\rangle)(\mathcal{O}_F/\pi^d)$. Let i be the largest integer such that $\phi(\overline{x}) \in \pi^i L/\pi^d L$. Then we must have $0 \leqslant i \leqslant [\frac{n}{2}]$ because $\nu_\pi(q(x)) = n$. Define $\widetilde{\phi} : \langle \pi^{-i}x\rangle/\pi^{d-i}\langle \pi^{-i}x\rangle \to L/\pi^{d-i}L$ to be $\widetilde{\phi}(\overline{\pi^{-i}x}) = \pi^{-i}\phi(\overline{x})$. Since we only know that $q(\phi(\overline{x})) \equiv q(x) \bmod \pi^d$, there exists an element $t \in \mathcal{O}_F/\pi^i$ such that

$$q\left(\pi^{-i}\phi(\overline{x})\right) \equiv \pi^{-2i}q(x) + \pi^{d-2i}t \equiv \pi^{-2i}q(x)\left(1 + \pi^d q(x)^{-1}t\right) \bmod \pi^{d-i}.$$

Hence $\widetilde{\phi} \in \mathrm{PRep}\left(L, \langle \pi^{-i}x\rangle[1 + \pi^d q(x)^{-1}t]\right)(\mathcal{O}_F/\pi^{d-i})$.

Since $d \geqslant n+1$, we know that the lattice $\langle \pi^{-i}x\rangle[1 + \pi^d q(x)^{-1}t]$ is isometric to $\langle \pi^{-i}x\rangle$, therefore

$$\bigsqcup_{t \in \mathcal{O}_F/(\pi^i)} \mathrm{PRep}\left(L, \langle \pi^{-i}x\rangle[1 + \pi^d q(x)^{-1}t]\right)(\mathcal{O}_F/\pi^{d-i}) \simeq \bigsqcup_{t \in \mathcal{O}_F/(\pi^i)} \mathrm{PRep}\left(L, \langle \pi^{-i}x\rangle\right)(\mathcal{O}_F/\pi^{d-i}).$$

□

Let $0 \leqslant i \leqslant [\frac{n}{2}]$ be an integer. Let $d \geqslant n+1$ be an integer. For an element $\phi \in \mathrm{PRep}\left(L, \langle \pi^{-i}x\rangle\right)$, we have

$$\phi(\overline{\pi^{-i}x}) = \overline{x_1} + \overline{x_2},$$

where $\overline{x_1} \in H_1[\pi]/\pi^{d-i}H_1[\pi]$ and $\overline{x_2} \in H_2/\pi^{d-i}H_2$. Define

$$\mathrm{PRep}_1(L, \langle \pi^{-i}x\rangle)(\mathcal{O}_F/\pi^{d-i}) := \{\phi \in \mathrm{PRep}\left(L, \langle \pi^{-i}x\rangle\right)(\mathcal{O}_F/\pi^{d-i}) : \overline{x_2} \in \pi H_2/\pi^{d-i}H_2\},$$
$$\mathrm{PRep}_2(L, \langle \pi^{-i}x\rangle)(\mathcal{O}_F/\pi^{d-i}) := \{\phi \in \mathrm{PRep}\left(L, \langle \pi^{-i}x\rangle\right)(\mathcal{O}_F/\pi^{d-i}) : \overline{x_2} \notin \pi H_2/\pi^{d-i}H_2\}.$$

Then we have a decomposition

$$\mathrm{PRep}\left(L, \langle \pi^{-i}x\rangle\right)(\mathcal{O}_F/\pi^{d-i}) = \mathrm{PRep}_1(L, \langle \pi^{-i}x\rangle)(\mathcal{O}_F/\pi^{d-i}) \bigsqcup \mathrm{PRep}_2(L, \langle \pi^{-i}x\rangle)(\mathcal{O}_F/\pi^{d-i}).$$

Lemma 3.2.4. *Let $0 \leqslant i \leqslant [\frac{n}{2}]$ be an integer. Let $d \geqslant n+1$ be an integer. We have*

$$\#\mathrm{PRep}_2(L, \langle \pi^{-i}x\rangle)(\mathcal{O}_F/\pi^{d-i}) = q^{n_1(d-i)} \cdot \#\mathrm{PRep}(H_{n_2}^{\varepsilon_2}, \langle \pi^{-i}x\rangle)(\mathcal{O}_F/\pi^{d-i}).$$
$$\#\mathrm{PRep}_1(L, \langle \pi^{-i}x\rangle)(\mathcal{O}_F/\pi^{d-i})$$
$$= \begin{cases} q^{n_2(d-i-1)+1} \cdot \#\mathrm{PRep}\left(H_{n_1}^{\varepsilon_1}, \langle \pi^{-i}x\rangle[\pi^{-1}]\right)(\mathcal{O}_F/\pi^{d-i}), & \text{if } \pi^{-2i-1}q(x) \in \mathcal{O}_F; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For simplicity, in the proof we denote $H_{n_1}^{\varepsilon_1}$ by H_1, denote $H_{n_2}^{\varepsilon_2}$ by H_2.

Let $\phi \in \mathrm{PRep}_1\left(L, \langle \pi^{-i}x\rangle\right)(\mathcal{O}_F/\pi^{d-i})$. Then we have

$$\phi(\overline{\pi^{-i}x}) = \overline{x_1} + \overline{\pi x_2'} \in L/\pi^{d-i}L,$$

where $\overline{x_1} \in H_1[\pi]/\pi^{d-i}H_1[\pi]$ and $\overline{x_1} \notin \pi H_1[\pi]/\pi^{d-i}H_1[\pi]$, and $\overline{x_2'} \in H_2/\pi^{d-i-1}H_2$. Therefore we get a map

$$\mathrm{pr}_2 : \mathrm{PRep}_1(L, \langle \pi^{-i}x\rangle)(\mathcal{O}_F/\pi^{d-i}) \to H_2/\pi^{d-i-1}H_2, \quad \phi \mapsto \overline{x_2'}.$$

Conversely, given an element $\overline{x_2'} \in H_2/\pi^{d-i-1}H_2$, the inverse image of $\overline{x_2'}$ under the map pr_2 is

$$\mathrm{pr}_2^{-1}(\overline{x_2'}) = \{\overline{x_1} \in H_1[\pi]/\pi^{d-i}H_1[\pi] : q_{H_1[\pi]}(\overline{x_1}) + q_{H_2}(\overline{\pi x_2'}) \equiv q(\pi^{-i}x) \bmod \pi^{d-i}$$
$$\text{and } \overline{x_1} \notin \pi H_1[\pi]/\pi^{d-i}H_1[\pi]\}.$$

For an element $\overline{x_1} \in \mathrm{pr}_2^{-1}(\overline{x_2'})$, we have $\pi q_{H_1}(\overline{x_1}) + \pi^2 q_{H_2}(\overline{x_2'}) \equiv \pi^{-2i}q(x) \bmod \pi^{d-i}$. Hence

$$q_{H_1}(\overline{x_1}) \equiv \pi^{-2i-1}q(x) - \pi q_{H_2}(\overline{x_2'}) \bmod \pi^{d-i-1}.$$

Therefore there exists an element $u \in \mathcal{O}_F/\pi$ such that

$$q_{H_1}(\overline{x_1}) \equiv \pi^{-2i-1}q(x) - \pi q_{H_2}(\overline{x_2'}) + \pi^{d-i-1}u \bmod \pi^{d-i},$$

then we get the following decomposition

$$\mathrm{pr}_2^{-1}(\overline{x_2'}) = \bigsqcup_{u \in \mathcal{O}_F/\pi} \{\overline{x_1} \in H_1/\pi^{d-i}H_1 : q_{H_1}(\overline{x_1}) \equiv \pi^{-2i-1}q(x) - \pi q_{H_2}(\overline{x_2'}) + \pi^{d-i-1}u \bmod \pi^{d-i},$$
$$\text{and } \overline{x_1} \notin \pi H_1/\pi^{d-i}H_1\}.$$

Denote by L_u the u-th term on the right hand side. Notice that $\pi^{-2i-1}q(x) - \pi q_{H_2}(\overline{x_2'}) + \pi^{d-i-1}u \equiv \pi^{-2i-1}q(x) \bmod \pi$. By the formula (12) and the definition of primitive local density, the primitive local density of a rank 1 quadratic lattice into a self-dual lattice only depends on the mod π reduction of the rank 1 lattice, we get

$$\#L_u = \begin{cases} \#\mathrm{PRep}(H_1, \langle \pi^{-i}x\rangle[\pi^{-1}])(\mathcal{O}_F/\pi^{d-i}), & \text{if } \pi^{-2i-1}q(x) \in \mathcal{O}_F; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore we have

$$\#\mathrm{pr}_2^{-1}(\overline{x_2'}) = \begin{cases} q \cdot \#\mathrm{PRep}\left(H_1, \langle \pi^{-i}x\rangle[\pi^{-1}]\right)(\mathcal{O}_F/\pi^{d-i}), & \text{if } \pi^{-2i-1}q(x) \in \mathcal{O}_F; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} \#\mathrm{PRep}_1(L, \langle \pi^{-i}x\rangle)(\mathcal{O}_F/\pi^{d-i}) &= \#H_2/\pi^{d-i-1}H_2 \cdot \#\mathrm{pr}_2^{-1}(\overline{x_2'}) \\ &= \begin{cases} q^{n_2(d-i-1)+1} \cdot \#\mathrm{PRep}\left(H_{n_1}^{\varepsilon_1}, \langle \pi^{-i}x\rangle[\pi^{-1}]\right)(\mathcal{O}_F/\pi^{d-i}), & \text{if } \pi^{-2i-1}q(x) \in \mathcal{O}_F; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Now we compute $\#\mathrm{PRep}_2(L, \langle \pi^{-i}x\rangle)(\mathcal{O}_F/\pi^{d-i})$. The idea is similar to the previous computations. Let $\phi \in \mathrm{PRep}_2\left(L, \langle \pi^{-i}x\rangle\right)(\mathcal{O}_F/\pi^{d-i})$. Then we have

$$\phi(\overline{\pi^{-i}x}) = \overline{x_1} + \overline{x_2} \in L/\pi^{d-i}L,$$

where $\overline{x_1} \in H_1[\pi]/\pi^{d-i}H_1[\pi]$, $\overline{x_2} \in H_2/\pi^{d-i}H_2$ and $\overline{x_2} \notin \pi H_2/\pi^{d-i}H_2$. Therefore we get a map

$$\mathrm{pr}_1 : \mathrm{PRep}_2(L, \langle \pi^{-i}x\rangle)(\mathcal{O}_F/\pi^{d-i}) \to H_1[\pi]/\pi^{d-i}H_1[\pi], \quad \phi \mapsto \overline{x_1}.$$

Conversely, given an element $\overline{x_1} \in H_1[\pi]/\pi^{d-i}H_1[\pi]$, the inverse image of $\overline{x_1}$ under the map pr_1 is

$$\begin{aligned} \mathrm{pr}_1^{-1}(\overline{x_1}) = \{\overline{x_2} \in H_2/\pi^{d-i}H_2 : q(\overline{x_2}) \equiv q(\pi^{-i}x) - q_{H_1[\pi]}(\overline{x_1}) \bmod \pi^{d-i}, \\ \text{and } \overline{x_2} \notin \pi H_2/\pi^{d-i}H_2\}. \end{aligned}$$

Notice that $q(\pi^{-i}x) - q_{H_1[\pi]}(\overline{x_1}) \equiv q(\pi^{-i}x) - \pi q_{H_1}(\overline{x_1}) \equiv q(\pi^{-i}x) \bmod \pi$. By the formula (12), we get

$$\#\mathrm{pr}_1^{-1}(\overline{x_1}) = \#\mathrm{PRep}\left(H_2, \langle \pi^{-i}x\rangle\right)(\mathcal{O}_F/\pi^{d-i}).$$

Therefore

$$\begin{aligned} \#\mathrm{PRep}_2(L, \langle \pi^{-i}x\rangle)(\mathcal{O}_F/\pi^{d-i}) &= \#H_1[\pi]/\pi^{d-i}H_1[\pi] \cdot \#\mathrm{PRep}\left(H_2, \langle \pi^{-i}x\rangle\right)(\mathcal{O}_F/\pi^{d-i}) \\ &= q^{n_1(d-i)} \cdot \#\mathrm{PRep}\left(H_{n_2}^{\varepsilon_2}, \langle \pi^{-i}x\rangle\right)(\mathcal{O}_F/\pi^{d-i}). \end{aligned}$$

□

3.2.2. *A restriction map.* Let d be a positive integer. We define a map $\mathrm{Rep}(L, M \oplus \langle x\rangle)(\mathcal{O}_F/\pi^d) \to \mathrm{Rep}(L, \langle x\rangle)(\mathcal{O}_F/\pi^d)$ as follows: for an element $\phi \in \mathrm{Rep}(L, M \oplus \langle x\rangle)(\mathcal{O}_F/\pi^d)$, define

$$\mathrm{res}(\phi)(\overline{x}) = \phi(\overline{x}).$$

Lemma 3.2.5. *Let $0 \leqslant i \leqslant [\frac{n}{2}]$ be an integer. When $d \geqslant n+1$ is a large enough integer,*

(a) *For an element $\phi \in \mathrm{PRep}_1(L, \langle \pi^{-i}x\rangle)(\mathcal{O}_F/\pi^{d-i})$, we have*

$$\#\mathrm{res}^{-1}(\phi) = \begin{cases} \#\mathrm{Rep}(H_{n_1-2}^{\varepsilon_1}[\pi] \oplus H_{n_2}^{\varepsilon_2} \oplus \langle \pi^{-i}x\rangle[-1], M)(\mathcal{O}_F/\pi^d) \cdot q^{(i+1)m}, & \text{if } n_1 \geqslant 3; \\ \#\mathrm{Rep}(H_{n_2}^{\varepsilon_2} \oplus \langle \pi^{-i}x\rangle[-\varepsilon_1], M)(\mathcal{O}_F/\pi^d) \cdot q^{(i+1)m}, & \text{if } n_1 = 2. \end{cases}$$

(b) *For an element* $\phi \in \mathrm{PRep}_2(L, \langle \pi^{-i}x \rangle)(\mathcal{O}_F/\pi^{d-i})$*, we have*

$$\#\mathrm{res}^{-1}(\phi) = \begin{cases} \#\mathrm{Rep}(H_{n_1}^{\varepsilon_1}[\pi] \oplus H_{n_2-2}^{\varepsilon_2} \oplus \langle \pi^{-i}x \rangle[-1], M)(\mathcal{O}_F/\pi^d) \cdot q^{im}, & \text{if } n_2 \geqslant 3; \\ \#\mathrm{Rep}(H_{n_1}^{\varepsilon_1} \oplus \langle \pi^{-i}x \rangle[-\varepsilon_2], M)(\mathcal{O}_F/\pi^d) \cdot q^{im}, & \text{if } n_2 = 2. \end{cases}$$

Proof. We only give the proof when $n_1, n_2 \geqslant 3$, the other cases where $n_1 = 2$ or $n_2 = 2$ can be proved by exactly the same strategy. Let $\{e_i\}_{i=1}^m$ be a basis of the lattice M. Let $j \in \{1, 2\}$ be an element.

Let $\phi \in \mathrm{PRep}_j(L, \langle \pi^{-i}x \rangle)(\mathcal{O}_F/\pi^{d-i})$ be an element. Let $\overline{\theta} = \phi(\overline{\pi^{-i}x}) \in L/\pi^{d-i}L$. Let $\theta \in L$ be a lift of $\overline{\theta}$ such that $q(\theta) = \pi^{-2i}q(x)$. Then

$$\mathrm{res}^{-1}(\phi) = \{(\overline{x_1}, \cdots, \overline{x_m}) \in (L/\pi^d L)^m : (\overline{x_k}, \overline{x_t}) \equiv (\overline{e_k}, \overline{e_t}) \bmod \pi^d, (\overline{x_k}, \overline{\pi^i\theta}) \equiv 0 \bmod \pi^d.\}.$$

By Lemma 3.1.1 and Lemma 3.1.2, we know that

$$L_{1,i} := \langle \theta \rangle^{\perp} \simeq H_{n_1-2}^{\varepsilon_1}[\pi] \oplus H_{n_2}^{\varepsilon_2} \oplus \langle \pi^{-i}x \rangle[-1], \tag{15}$$

$$L_{2,i} := \langle \theta \rangle^{\perp} \simeq H_{n_1}^{\varepsilon_1}[\pi] \oplus H_{n_2-2}^{\varepsilon_2} \oplus \langle \pi^{-i}x \rangle[-1]. \tag{16}$$

For simplicity, we still use L_j to denote the lattice $L_{j,i}$. We have an exact sequence

$$0 \longrightarrow L_j \oplus \langle \theta \rangle \xrightarrow{i_j} L \longrightarrow Q_j \longrightarrow 0. \tag{17}$$

The quotient Q_j is a finite group since $L_j \oplus \langle \theta \rangle$ and L have the same rank. Tensoring the sequence (17) by $\mathcal{O}_F/\pi^d$ for a sufficiently large integer d, we get

$$0 \longrightarrow K_j \longrightarrow L_j/\pi^d L_j \oplus \langle \theta \rangle/\pi^d \langle \theta \rangle \xrightarrow{\overline{i_j}} L/\pi^d L \longrightarrow Q_j \longrightarrow 0,$$

where $\#K_j = \#Q_j = q^{n-2i+j-2}$. Let $\overline{i_j}^m = \overline{i_j} \times \cdots \times \overline{i_j}$.

Claim: When d is large enough, the map $(\overline{i_j}^m)^{-1}(\mathrm{res}^{-1}(\phi)) \to \mathrm{res}^{-1}(\phi)$ is surjective.

Proof of the claim: Let $(\overline{x_1}, \cdots, \overline{x_m}) \in \mathrm{res}^{-1}(\phi)$. Let $x_1, \cdots, x_m \in L$ be lifts of the elements $\overline{x_1}, \cdots, \overline{x_m} \in L/\pi^d L$. Then for all integers $1 \leqslant i \leqslant m$,

$$x_i = \frac{(x_i, \theta)}{2q(\theta)} \cdot \theta + x_i',$$

where $(x_i', \theta) = 0$. When d is large enough, the element $\frac{(x_i,\theta)}{2q(\theta)} \in \mathcal{O}_F$. Hence $x_i' \in L$. Therefore $x_i' \in \langle \theta \rangle^{\perp} = L_j$. Hence

$$(\overline{x_1}, \cdots, \overline{x_m}) = \left(\overline{\frac{(x_i, \theta)}{2q(\theta)} \cdot \theta}, \cdots, \overline{\frac{(x_m, \theta)}{2q(\theta)} \cdot \theta} \right) + \left(\overline{x_1'}, \cdots, \overline{x_m'} \right) \in \mathrm{Im}(\overline{i_j}^m).$$

■

Now we fix an isomorphism of $\mathcal{O}_F$-modules $\langle \theta \rangle \simeq \mathcal{O}_F$ such that θ is mapped to 1. Then when d is large enough,

$$(\overline{i_j}^m)^{-1}(\mathrm{res}^{-1}(\phi)) = \{(\overline{x_1'}, \cdots, \overline{x_m'}) \in (L_j/\pi^d L_j)^m, (\overline{a_1}, \cdots, \overline{a_m}) \in (\mathcal{O}_F/\pi^d)^m : \overline{a_k} \in \pi^{d-n+i}/\pi^d,$$
$$(\overline{x_k'}, \overline{x_t'}) \equiv (\overline{e_k}, \overline{e_t}) \bmod \pi^d.\}$$

Therefore $(\overline{i_j}^m)^{-1}(\mathrm{res}^{-1}(\phi)) \simeq \mathrm{Rep}(L_j, M)(\mathcal{O}_F/\pi^d) \times (\mathcal{O}_F/\pi^{n-i})^m$. Hence

$$\#\mathrm{res}^{-1}(\phi) = \frac{\#(\overline{i_j}^m)^{-1}(\mathrm{res}^{-1}(\phi))}{\#K_j^m} = \#\mathrm{Rep}(L_j, M)(\mathcal{O}_F/\pi^d) \cdot q^{(i+2-j)m}$$

□

Proof of Theorem 3.2.1. Let $d \geqslant n+1$ be a large enough integer, we have

$$\begin{aligned} \#\mathrm{Rep}(L, M \oplus \langle x \rangle)(\mathcal{O}_F/\pi^d) = \sum_{i=0}^{[n/2]} & q^i \cdot \#\mathrm{PRep}_1(L, \langle \pi^{-i}x \rangle)(\mathcal{O}_F/\pi^{d-i}) \cdot \#\mathrm{res}^{-1}(\phi_{1,i}) \\ & + q^i \cdot \#\mathrm{PRep}_2(L, \langle \pi^{-i}x \rangle)(\mathcal{O}_F/\pi^{d-i}) \cdot \#\mathrm{res}^{-1}(\phi_{2,i}) \end{aligned} \tag{18}$$

where $\phi_{1,i} \in \mathrm{PRep}_1(L, \langle \pi^{-i}x \rangle)$ and $\phi_{2,i} \in \mathrm{PRep}_2(L, \langle \pi^{-i}x \rangle)$. By the definitions of $L_{1,i}$ (15), $L_{2,i}$ (16) and (18), we have

$$\begin{aligned} \mathrm{Den}(L, M \oplus \langle x \rangle) = \sum_{i=0}^{[n/2]} & q^{m+1-n_2} \cdot q^{i(m+2-n_1-n_2)} \cdot \mathrm{Den}(L_{1,i}, M) \cdot \mathrm{Pden}(H_{n_1}^{\varepsilon_1}, \langle \pi^{-i}x \rangle[\pi^{-1}]) \\ & + q^{i(m+2-n_1-n_2)} \cdot \mathrm{Den}(L_{2,i}, M) \cdot \mathrm{Pden}(H_{n_2}^{\varepsilon_2}, \langle \pi^{-i}x \rangle). \end{aligned}$$

Notice that $L_{1,0} = L_1$ and $L_{2,0} = L_2$. Therefore

$$\begin{aligned} &\mathrm{Den}\left(L, M \oplus \langle x \rangle\right) - q^{m+2-n_1-n_2} \cdot \mathrm{Den}\left(L, M \oplus \langle \pi^{-1}x \rangle\right) \\ = & q^{m+1-n_2} \cdot \mathrm{Den}\left(L_1, M\right) \cdot \mathrm{Pden}\left(H_{n_1}^{\varepsilon_1}, \langle x \rangle[\pi^{-1}]\right) + \mathrm{Den}\left(L_2, M\right) \cdot \mathrm{Pden}\left(H_{n_2}^{\varepsilon_2}, \langle x \rangle\right). \end{aligned}$$

3.3. **Analytic difference formula.** Let $s \in F$ be a nonzero number. $H_0(s)$ be the following lattice over $\mathcal{O}_F$ of rank 4,

$$H_0(s) = \left\{ \begin{pmatrix} a & b \\ sc & d \end{pmatrix} : a, b, c, d \in \mathcal{O}_F \right\}.$$

We equip the lattice $H_0(s)$ with the quadratic form given by the determinant morphism det : $H_0(s) \to F$. Then

$$H_0(s) \simeq H_2^+[s] \oplus H_2^+.$$

Let $H_{n_1}^{\varepsilon_1} = H_2^+$ and $H_{n_2}^{\varepsilon_2} = H_{2k+2}^+$. Applying Theorem 3.2.1 and the formula (12), we get the following formula about $\mathrm{Den}(X, H_0(\pi), L)$ which we defined in (5).

Lemma 3.3.1. *Let $\langle x\rangle$ be a rank 1 quadratic lattice generated by x such that $n := \nu_\pi(q(x)) \geqslant 0$. Let $L^\flat$ be a rank 2 quadratic lattice over $\mathcal{O}_F$. Then*

$$\mathrm{Den}\left(X, H_0(\pi), L^\flat \obot \langle x\rangle\right) - X^2 \cdot \mathrm{Den}\left(X, H_0(\pi), L^\flat \obot \langle \pi^{-1}x\rangle\right)$$
$$= \begin{cases} \left(1-q^{-1}X\right)\cdot \mathrm{Den}\left(X, \langle x\rangle[-1]\obot H_2^+[\pi], L^\flat\right), & \text{if } n=0;\\[2ex] \left(1-q^{-1}X\right)(1+X)\cdot \mathrm{Den}\left(X, \langle x\rangle[-1]\obot H_2^+[\pi], L^\flat\right) \\ \qquad +(q-1)X^2\cdot \mathrm{Den}\left(X, \langle x\rangle[-1]\obot H_2^+, L^\flat\right), & \text{if } n=1;\\[2ex] \left(1-q^{-1}X\right)(1+X)\cdot \mathrm{Den}\left(X, \langle x\rangle[-1]\obot H_2^+[\pi], L^\flat\right) \\ \qquad +2(q-1)X^2\cdot \mathrm{Den}\left(X, \langle x\rangle[-1]\obot H_2^+, L^\flat\right), & \text{if } n\geqslant 2.\end{cases}$$

Definition 3.3.2. Let L and $L^\flat$ be quadratic lattices of rank 3 and 2 over $\mathcal{O}_F$ respectively. Let $\langle x\rangle$ be a rank 1 quadratic lattice generated by x. Define the (normalized) *derived local densities*

$$\partial\mathrm{Den}\left(H_0(\pi), L\right) := -2\cdot \frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \frac{\mathrm{Den}(X, H_0(\pi), L)}{\mathrm{Den}(H_0(\pi), H_2^+\obot H_1^+[\pi])},$$
$$\partial\mathrm{Den}\left(H_0(\pi)^\vee, L\right) := -2\cdot \frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \frac{\mathrm{Den}(X, H_0(\pi)^\vee, L)}{\mathrm{Den}(H_0(\pi)^\vee, H_2^+\obot H_1^+[\pi^{-1}])},$$
$$\partial\mathrm{Den}\left(\langle x\rangle[-1]\obot H_2^+, L^\flat\right) := -\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \frac{\mathrm{Den}(X, \langle x\rangle[-1]\obot H_2^+, L^\flat)}{\mathrm{Den}(H_2^+, H_1^+)},$$
$$\partial\mathrm{Den}\left(\langle x\rangle[-1]\obot H_2^+[\pi], L^\flat\right) := -\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \frac{\mathrm{Den}(X, \langle x\rangle[-1]\obot H_2^+[\pi], L^\flat)}{\mathrm{Den}(H_2^+[\pi], H_1^+[\pi])}.$$

Lemma 3.3.3. *Let $\mathbb{B}$ be the unique division quaternion algebra over F. Let $L^\flat \subset \mathbb{B}$ be an $\mathcal{O}_F$-lattice of rank 2. Let $x\in\mathbb{B}$ be an element such that $n := \nu_\pi(q(x))\geqslant 0$ and $x \perp L^\flat$. Then*

$$\partial\mathrm{Den}\left(H_0(\pi), L^\flat\obot\langle x\rangle\right) - \partial\mathrm{Den}\left(H_0(\pi), L^\flat\obot\langle\pi^{-1}x\rangle\right)$$
$$= \begin{cases} \partial\mathrm{Den}\left(\langle x\rangle[-1]\obot H_2^+[\pi], L^\flat\right), & \text{if } n=0;\\[2ex] 2\cdot\partial\mathrm{Den}\left(\langle x\rangle[-1]\obot H_2^+[\pi], L^\flat\right) \\ \quad +\partial\mathrm{Den}\left(\langle x\rangle[-1]\obot H_2^+, L^\flat\right), & \text{if } n=1;\\[2ex] 2\cdot\partial\mathrm{Den}\left(\langle x\rangle[-1]\obot H_2^+[\pi], L^\flat\right) \\ +2\cdot\partial\mathrm{Den}\left(\langle x\rangle[-1]\obot H_2^+, L^\flat\right), & \text{if } n\geqslant 2.\end{cases}$$

Proof. Notice that $\mathrm{Den}(H_2^+, H_1^+) = 1 - q^{-1}$, therefore we have

$$\begin{aligned}\mathrm{Den}(H_0(\pi), H_2^+ \obot H_1^+[\pi]) &= \mathrm{Pden}(H_2^+, H_1^+) \cdot \mathrm{Den}(H_1^+ \obot H_2^+[\pi], H_1^+ \obot H_1^+[\pi]) \\ &= (1-q^{-1}) \cdot \mathrm{Pden}(H_1^+, H_1^+) \cdot \mathrm{Den}(H_2^+[\pi], H_1^+[\pi]) \\ &= 2(1-q^{-1}) \cdot q \cdot (1-q^{-1}) = 2q^{-1}(q-1)^2.\end{aligned}$$

Hence $\mathrm{Den}(H_0(\pi)^\vee, H_2^+ \obot H_1^+[\pi^{-1}]) = q^{-6} \cdot \mathrm{Den}(H_0(\pi), H_2^+ \obot H_1^+[\pi]) = 2q^{-7}(q-1)^2$. We also have $\mathrm{Den}(H_2^+[\pi], H_1^+[\pi]]) = q \cdot \mathrm{Den}(H_2^+, H_1^+) = q - 1$.

The lattice $L^\flat \obot \langle x\rangle$ is contained in the space $\mathbb{B}$, therefore it can't be isometrically embedded into the space H_4^+ by [Kud97, (I.14)]. Notice that the lattice $H_0(\pi)$ is contained in the space H_4^+, hence we have

$$\mathrm{Den}\left(1, H_0(\pi), L^\flat \obot \langle x\rangle\right) = \mathrm{Den}\left(1, H_0(\pi), L^\flat \obot \langle \pi^{-1}x\rangle\right) = 0.$$

Taking derivatives and evaluate at $X = 1$ of both sides of the equation in Lemma 3.3.1, we get the desired formula. □

3.4. **Two identities.** In §3.4 and §3.5, we assume $F = \mathbb{Q}_p$ since we need to use a formula of Yang [Yan98] which is over $\mathbb{Q}_p$. Although the argument should work for a general p-adic field F, we keep this assumption for safety.

Lemma 3.4.1. *Let $\mathbb{B}$ be the unique division quaternion algebra over $\mathbb{Q}_p$. Let $L^\flat \subset \mathbb{B}$ be an $\mathbb{Z}_p$-lattice of rank 2. Let $x \in \mathbb{B}$ be an element such that $x \perp L^\flat$ and $\nu_p(q(x)) \geqslant \max\{\max(L^\flat), 2\}$. Then*

$$\partial\mathrm{Den}\left(\langle x\rangle \obot H_2^+[p], L^\flat\right) = p^2 \cdot \partial\mathrm{Den}\left(\langle x\rangle[p^{-1}] \obot H_2^+, L^\flat[p^{-1}]\right) - 1.$$

Proof. Since $L^\flat$ has rank 2, we can apply the results of Yang [Yan98], which gives a precise formula for $\mathrm{Den}(X, M, L)$ where L is a lattice of rank 2 and M is a lattice of rank $m > 2$. Let $0 \leqslant a \leqslant b$ be the fundamental invariants of the quadratic lattice $L^\flat$. By direct computations using [Yan98], we get

$$\partial\mathrm{Den}\left(\langle x\rangle \obot H_2^+[p], L^\flat\right) = \frac{1}{(p-1)^2} \cdot \begin{cases} ap^{(a+b+6)/2} - ap^{(a+b+2)/2} - 2p^{a+2} + p^2 + 2p - 1, & \text{if } a \equiv b \bmod 2; \\ ap^{(a+b+5)/2} - ap^{(a+b+3)/2} - 2p^{a+2} + p^2 + 2p - 1, & \text{if } a \not\equiv b \bmod 2.\end{cases}$$

And

$$\partial\mathrm{Den}\left(\langle x\rangle[p^{-1}] \obot H_2^+, L^\flat[p^{-1}]\right) = \frac{1}{(p-1)^2} \cdot \begin{cases} ap^{(a+b+2)/2} - ap^{(a+b-2)/2} - 2p^{a} + 2, & \text{if } a \equiv b \bmod 2; \\ ap^{(a+b+1)/2} - ap^{(a+b-1)/2} - 2p^{a} + 2, & \text{if } a \not\equiv b \bmod 2.\end{cases}$$

The identity in the lemma can be verified by comparing the above two formulas. □

Lemma 3.4.2. *Let $\mathbb{B}$ be the unique division quaternion algebra over $\mathbb{Q}_p$. Let $L^\flat \subset \mathbb{B}$ be an $\mathbb{Z}_p$-lattice of rank 2. Let $x \in \mathbb{B}$ be an element such that $x \perp L^\flat$ and $\nu_p(q(x)) \geqslant \max\{\max(L^\flat), 1\}$. Then*

$$\begin{aligned}&\partial\mathrm{Den}\left(H_0(p)^\vee, L^\flat \obot \langle x\rangle\right) - \partial\mathrm{Den}\left(H_0(p)^\vee, L^\flat \obot \langle p^{-1}x\rangle\right) \\ &= \partial\mathrm{Den}\left(H_0(p), L^\flat[p] \obot \langle x\rangle[p]\right) - \partial\mathrm{Den}\left(H_0(p), L^\flat[p] \obot \langle p^{-1}x\rangle[p]\right).\end{aligned}$$

Proof. Let $-1 \leqslant a \leqslant b$ be the fundamental invariants of the quadratic lattice $L^\flat$. In this case, we have

$$\begin{aligned}
&\mathrm{Den}(H_0(p)^\vee \oplus H_{2k}^+, L^\flat \oplus \langle x\rangle) - p^{-2k}\cdot \mathrm{Den}(H_0(p)^\vee \oplus H_{2k}^+, L^\flat \oplus \langle p^{-1}x\rangle)\\
=&p^{-6}\left(\mathrm{Den}\left(H_0(p)\oplus H_{2k}^+[p], L^\flat[p]\oplus\langle x\rangle[p]\right) - p^{-2k}\cdot \mathrm{Den}\left(H_0(p)\oplus H_{2k}^+[p], L^\flat[p]\oplus\langle p^{-1}x\rangle[p]\right)\right)\\
=&p^{-6}\Bigg(p\cdot \mathrm{Pden}\left(H_{2k+2}^+, \langle x\rangle\right)\cdot \mathrm{Den}\left(H_{2k}^+[p]\oplus H_2^+\oplus\langle x\rangle[-p], L^\flat[p]\right)\\
&\qquad\qquad + \mathrm{Pden}\left(H_2^+, \langle x\rangle[p]\right)\cdot \mathrm{Den}\left(H_{2k+2}^+[p]\oplus\langle x\rangle[-p], L^\flat[p]\right)\Bigg).
\end{aligned}$$

Let $f(X), f_1(X), f_2(X)$ be polynomials such that for all $k \geqslant 0$,

$$\begin{aligned}
f(p^{-k}) &= \mathrm{Den}\left(H_2^+[p]\oplus\langle x\rangle[-p]\oplus H_{2k}^+, L^\flat[p]\right),\\
f_1(p^{-k}) &= \mathrm{Den}\left(H_{2k}^+[p]\oplus H_2^+\oplus\langle x\rangle[-p], L^\flat[p]\right),\\
f_2(p^{-k}) &= \mathrm{Den}\left(H_{2k+2}^+[p]\oplus\langle x\rangle[-p], L^\flat[p]\right).
\end{aligned}$$

By the main theorem of [Yan98], there exists two polynomials $R_1(X), R_2(X)$ such that

$$f(X) = 1 + R_1(X) + R_2(X),$$
$$f_1(X) = 1 + p^{-1}X^{-1}R_1(X) + p^{-2}X^{-2}R_2(X),\ \ f_2(X) = 1 + X^{-1}R_1(X) + X^{-2}R_2(X).$$

Then we conclude that

$$\begin{aligned}
&\partial\mathrm{Den}\left(H_0(p)^\vee, L^\flat\oplus\langle x\rangle\right) - \partial\mathrm{Den}\left(H_0(p)^\vee, L^\flat\oplus\langle p^{-1}x\rangle\right)\\
=&\,2p^{-1}\left((p+1)\cdot\partial\mathrm{Den}\left(\langle x\rangle[-p]\oplus H_2^+[p], L^\flat[p]\right) - R_1'(1)\right).
\end{aligned}$$

The number $R_1'(1)$ is given by the following explicit formula

$$R_1'(1) = \frac{1}{p-1}\cdot\begin{cases}(1+p)(1-p^{(a+b)/2+2}), & \text{if } a\equiv b \bmod 2;\\ 1+p-2p^{(a+b+1)+2}, & \text{if } a\not\equiv b \bmod 2.\end{cases}$$

Then the identity in the lemma can be verified by combining Lemma 3.3.3 and computations of $\partial\mathrm{Den}\left(\langle x\rangle[-p]\oplus H_2^+[p], L^\flat[p]\right)$ in the proof of Lemma 3.4.1. □

Remark 3.4.3. The assumption $p > 2$ is also used in this section due to the reliance on [Yan98]. This is the only essential use of $p > 2$ on the analytic side.

3.5. Base cases: the analytic side.

Lemma 3.5.1. *Suppose $F = \mathbb{Q}_p$. Let $\varepsilon \in \mathbb{Z}_p^\times$ be an element. Then*

$$\begin{aligned}
&\mathrm{Den}\left(X, H_0(p), H_2^-\oplus H_1^+[\varepsilon p]\right) = \left(1-p^{-1}X\right)\left(1+\varepsilon X\right)\left(1+\varepsilon(p-1)X - X^2\right);\\
&\mathrm{Den}\left(X, H_0(p), H_1^+[\varepsilon]\oplus H_2^-[p]\right) = \left(1-p^{-1}X\right)\left(1-2X^2+X^4\right);\\
&\mathrm{Den}\left(X, H_0(p)^\vee, H_1^+[\varepsilon p^{-1}]\oplus H_2^-\right) = p^{-7}(p-1)\cdot\left(p-(1+p)X+X^2\right);\\
&\mathrm{Den}\left(X, H_0(p)^\vee, H_2^-[p^{-1}]\oplus H_1^+[\varepsilon]\right) = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&\partial\mathrm{Den}\left(H_0(p), H_2^- \oplus H_1^+[\varepsilon p]\right) = -1;\\
&\partial\mathrm{Den}\left(H_0(p), H_1^+[\varepsilon] \oplus H_2^-[p]\right) = 0;\\
&\partial\mathrm{Den}\left(H_0(p)^\vee, H_2^-[p^{-1}] \oplus H_1^+[\varepsilon]\right) = 0;\\
&\partial\mathrm{Den}\left(H_0(p), H_1^+[\varepsilon p^{-1}] \oplus H_2^-\right) = 1.
\end{aligned}$$

Proof. By Lemma 3.3.1, we have that for positive integers k,

$$\begin{aligned}
\mathrm{Den}\left(p^{-k}, H_0(p), H_2^\varepsilon \oplus H_1^+[\varepsilon p]\right) &= (1-p^{-1-k})(1+p^{-k}) \cdot \mathrm{Den}(H_{2k}^+, H_2^\varepsilon)\\
&\quad + p^{1-2k}(1-p^{-1}) \cdot \mathrm{Den}(H_{2k+2}^+, H_2^\varepsilon).
\end{aligned}$$

By the calculations of $\mathrm{Den}(H_m^\varepsilon, H_n^\varepsilon)$ in [LZ22b, Definition 3.4.1, Definition 3.5.1], we have

$$\mathrm{Den}(H_{2k+2}^+, H_2^\varepsilon) = (1-p^{-k-1})(1+\varepsilon p^{-k}).$$

Then we get the first formula.

For the second formula, we have the following decomposition by Lemma 3.3.1,

$$\mathrm{Den}(H_0(p) \oplus H_{2k}^+, H_1^+[\varepsilon] \oplus H_2^-[p]) = \mathrm{Den}(H_0(p) \oplus H_{2k}^+, H_1^+[\varepsilon]) \cdot \mathrm{Den}(H_2^+[p] \oplus H_{2k}^{-\varepsilon}, H_2^-[p]).$$

Both local densities $\mathrm{Den}(H_0(p) \oplus H_{2k}^+, H_1^+[\varepsilon])$ and $\mathrm{Den}(H_2^+[p] \oplus H_{2k}^{-\varepsilon}, H_2^-[p])$ can be computed by the formulas in [Yan98].

For the third formula, notice that

$$\mathrm{Den}\left(X, H_0(p)^\vee, H_1^+[\varepsilon p^{-1}] \oplus H_2^-\right)|_{X=p^{-k}} = p^{-6}\mathrm{Den}\left(H_0(p) \oplus H_{2k}^+[p], H_1^+[\varepsilon] \oplus H_2^-[p]\right).$$

Again we can use Lemma 3.3.1 to reduce the local densities to the cases where formulas in [Yan98] can be applied.

For the last one, we notice that H_2^- can be mapped isometrically to $H_0(p)$, which implies that $H_2^-[p^{-1}]$ can't be mapped isometrically to $H_0(p)^\vee \oplus H_{2k}^+$ for all $k \geqslant 0$, hence the local density is identically zero. □

Part 2. Geometric side

4. Rapoport–Zink spaces and special cycles

From now on, we assume $F = \mathbb{Q}_p$. For simplicity, let $W = \breve{\mathbb{Z}}_p$. Let $\mathbb{B}$ be the unique division quaternion algebra over $\mathbb{Q}_p$. Let $\mathbb{X}$ be the unique (up to isomorphism) formal group of dimension 1 and height 2 over $\mathbb{F}$ with a principal polarization $\lambda : \mathbb{X} \to \mathbb{X}^\vee$.

4.1. **Rapoport–Zink spaces with hyperspecial level structures.** Let $\mathcal{N}_0$ be the following functor on the category Nilp_W: for any $S \in \mathrm{Nilp}_W$, the set $\mathcal{N}_0(S)$ is the isomorphism classes of pairs (X, ρ), where X is a p-divisible group over S and ρ is a height 0 quasi-isogeny between p-divisible groups $\rho : \mathbb{X} \times_{\mathbb{F}} \overline{S} \to X \times_S \overline{S}$. It is well-known that the functor $\mathcal{N}_0$ is represented by the formal scheme $\mathrm{Spf}\, W[[t]]$ over $\mathrm{Spf}\, W$ (see [VGW+07, Theorem 3.8, §7] for example).

Let $(X^{\mathrm{univ}}, \rho^{\mathrm{univ}})$ be the universal p-divisible group over the formal scheme $\mathcal{N}_0$. Let $\mathbb{D}(X^{\mathrm{univ}})$ be the (covariant)-Dieudonne crystal of the p-divisible group X^{univ} (see [Kim18, §2.3] for the construction). It is a locally free $\mathcal{O}_{\mathcal{N}_0}^{\mathrm{crys}}$-module crystal of rank 2. Given a morphism $S \to \mathcal{N}_0$ where S is an object in Nilp_W, let $\mathbb{D}(X^{\mathrm{univ}})_S$ be the pullback of the crystal $\mathbb{D}(X^{\mathrm{univ}})$ to the site $\mathrm{NCRIS}_W(S/\mathrm{Spec}\, W)$.

Let $\mathbb{D}(\mathbb{X})$ be the Dieudonne module of the p-divisible group $\mathbb{X}$. There exists a basis $[e_1, e_2]$ of the rank 2 free W-module $\mathbb{D}(\mathbb{X})$ such that the Hodge filtration on $\mathbb{D}(\mathbb{X})_{\mathbb{F}} := \mathbb{D}(\mathbb{X}) \otimes_W \mathbb{F}$ is given by

$$0 \to \mathrm{Fil}^1 \mathbb{D}(\mathbb{X})_{\mathbb{F}} = \mathbb{F} \cdot \overline{e_2} \to \mathbb{D}(\mathbb{X})_{\mathbb{F}}.$$

Adjusting the element t by some invertible element in the local ring $\mathcal{O}_{\mathcal{N}_0}$, the Hodge filtration on the rank 2 free $\mathcal{O}_{\mathcal{N}_0}$-module $\mathbb{D}(X^{\mathrm{univ}})(\mathcal{N}_0)$ is given by

$$0 \to \mathrm{Fil}^1 \mathbb{D}(X^{\mathrm{univ}})(\mathcal{N}_0) = \mathcal{O}_{\mathcal{N}_0} \cdot (e_2 + te_1) \to \mathbb{D}(X^{\mathrm{univ}})(\mathcal{N}_0).$$

4.2. **CM cycles: the hyperspecial case.** Recall we use $(X^{\mathrm{univ}}, \rho^{\mathrm{univ}})$ to denote the universal p-divisible group over the formal scheme $\mathcal{N}_0$. Let $\mathbb{B}^0$ be the subgroup of $\mathbb{B}$ consisting of trace 0 elements.

Definition 4.2.1. For any subset $H \subset \mathbb{B}^0$, define the CM cycle $\mathcal{Z}_{\mathcal{N}_0}(H) \subset \mathcal{N}_0$ to be the closed formal subscheme cut out by the condition,

$$\rho^{\mathrm{univ}} \circ x \circ (\rho^{\mathrm{univ}})^{-1} \in \mathrm{Hom}(X^{\mathrm{univ}}, X^{\mathrm{univ}}).$$

for all $x \in H$.

4.3. **Special cycles on the product: the hyperspecial case.** Let $\mathcal{N} = \mathcal{N}_0 \times_W \mathcal{N}_0$. It is a formal scheme which parameterizes two pairs $((X, \rho), (X', \rho'))$. Let $\big((X^{\mathrm{univ}}, \rho^{\mathrm{univ}}), (X'^{\mathrm{univ}}, \rho'^{\mathrm{univ}})\big)$ be the universal pairs over the formal scheme $\mathcal{N}$.

Definition 4.3.1. For any subset $H \subset \mathbb{B}$, define the special cycle $\mathcal{Z}_{\mathcal{N}}(H) \subset \mathcal{N}$ to be the closed formal subscheme cut out by the condition,

$$\rho'^{\mathrm{univ}} \circ x \circ (\rho^{\mathrm{univ}})^{-1} \in \mathrm{Hom}(X^{\mathrm{univ}}, X'^{\mathrm{univ}}).$$

for all $x \in H$.

Lemma 4.3.2. *Let $x \in \mathbb{B}$ be a nonzero element such that $q(x) \in \mathbb{Z}_p$. Then $\mathcal{Z}_{\mathcal{N}}(x)$ is a Cartier divisor on $\mathcal{N}$ (i.e., defined by one nonzero equation) and flat over W (i.e., the equation is not divisible by p). Moreover, let $x, y \in \mathbb{B}$ be two linearly independent elements, then the two divisors $\mathcal{Z}_{\mathcal{N}}(x)$ and $\mathcal{Z}_{\mathcal{N}}(y)$ intersect properly, and the irreducible components of the intersection $\mathcal{Z}_{\mathcal{N}}(x) \cap \mathcal{Z}_{\mathcal{N}}(y)$ are of the form $\mathrm{Spf}\, W_s$ where W_s is the ring of definition of a quasi-canonical lifting of level s.*

Proof. This is proved by Gross–Keating (see [GK93, (5.10)]). □

For $H = \{x\}$, we denote by $\mathcal{Z}_{\mathcal{N}}(x)$ the cycle $\mathcal{Z}_{\mathcal{N}}(\{x\})$. By the moduli interpretation of the special cycle $\mathcal{Z}_{\mathcal{N}}(x)$, there is a closed immersion $\mathcal{Z}_{\mathcal{N}}(p^{-1}x) \to \mathcal{Z}_{\mathcal{N}}(x)$.

Definition 4.3.3. Let $x \in \mathbb{B}$ be a nonzero element. Define the difference divisor associated to x on $\mathcal{N}$ to be the following effective Cartier divisor on the formal scheme $\mathcal{N}$,

$$\mathcal{D}_{\mathcal{N}}(x) = \mathcal{Z}_{\mathcal{N}}(x) - \mathcal{Z}_{\mathcal{N}}(p^{-1}x).$$

Remark 4.3.4. Terstiege first introduced the difference divisors on the unitary Rapoport–Zink spaces with hyperspecial level [Ter13a] and proved the regularity of them [Ter13b]. He also give the construction of difference divisors on the Rapoport–Zink space associated to a rank 4 self-dual quadratic lattice over $\mathbb{Z}_p$ other than $\mathrm{M}_2(\mathbb{Z}_p)$ and studied the intersection numbers of them in [Ter11], where he also proved the regularity of these difference divisors.

The formal scheme $\mathcal{N}$ is the (connected) Rapoport–Zink space associated to the self-dual lattice $\mathrm{M}_2(\mathbb{Z}_p)$ with quadratic form given by determinant. The difference divisors $\mathcal{D}_{\mathcal{N}}(x)$ are regular formal schemes (cf. [Zhu25, Theorem 6.2.2], see also [Zhu24b]), it is formally smooth over W if and only if $\nu_p(q(x)) = 0$. It's easy to see that $\mathcal{Z}_{\mathcal{N}}(1) \simeq \mathcal{N}_0$ and for an element $y \in \mathbb{B}^0 = \{x \in \mathbb{B} \mid \mathrm{tr}(x) = 0\}$, we have the following isomorphism:

$$\mathcal{Z}_{\mathcal{N}_0}(y) \simeq \mathcal{Z}_{\mathcal{N}}(y) \cap \mathcal{Z}(1).$$

Here we regard $\mathbb{B}^0$ as the special quasi-endomorphism space for $\mathcal{N}_0$. Let $\mathcal{D}_{\mathcal{N}_0}(y) = \mathcal{Z}_{\mathcal{N}_0}(y) - \mathcal{Z}_{\mathcal{N}_0}(p^{-1}y)$. It is a regular divisor on the formal scheme $\mathcal{N}_0$. It is isomorphic to the ring of definition of a (quasi-)canonical lifting of $\mathbb{X}$ and formally smooth over W if and only if $\nu_p(q(y)) = 0$.

4.4. **Rapoport–Zink spaces with cyclic level structures.** We have an isomorphism

$$\iota : \mathbb{B} \simeq \mathrm{End}^\circ(\mathbb{X}) := \mathrm{End}(\mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{19}$$

Let $x \mapsto \overline{x}$ be the main involution of the quaternion algebra $\mathbb{B}$. Let $x \mapsto x^\vee$ be the Rosati involution on $\mathrm{End}^\circ(\mathbb{X})$ induced by the principal polarization λ. The two involutions are identified under any choice of the isomorphism (19).

On the algebra $\mathbb{B}$, denote by q the quadratic form $q(x) = x \cdot \overline{x} \in \mathbb{Q}_p$. On the algebra $\mathrm{End}^\circ(\mathbb{X})$, denote by q_λ the quadratic form $q_\lambda(x) = x \circ x^\vee \in \mathbb{Q}_p$. The isomorphism ι is an isometry between the quadratic spaces $(\mathbb{B}, q)$ and $(\mathrm{End}^\circ(\mathbb{X}), q_\lambda)$. Moreover, the maximal order $\mathcal{O}_{\mathbb{B}}$ of $\mathbb{B}$ is mapped isometrically to $\mathrm{End}(\mathbb{X})$.

For all $x \in \mathbb{B}$, we consider the following contravariant set-valued functor $\mathcal{N}_0(x)$ defined over Nilp_W: for every $S \in \mathrm{Nilp}_W$, the set $\mathcal{N}_0(x)(S)$ consists of the isomorphism classes of elements of the following form $(X \xrightarrow{\pi} X', (\rho, \rho'))$, where

(a) $((X, \rho), (X', \rho')) \in \mathcal{N}(S)$;
(b) $\pi : X \to X'$ is a cyclic isogeny (i.e., $\ker(x)$ is a cyclic group scheme over S in the sense of [KM85, §6.1]) lifting $\rho' \circ x \circ \rho^{-1}$.

There is a natural morphism st_x from $\mathcal{N}_0(x)$ to $\mathcal{N}$ given as follows,

$$\begin{aligned} \mathrm{st}_x : \mathcal{N}_0(x) &\longrightarrow \mathcal{N}; \\ (X \xrightarrow{\pi} X', (\rho, \rho')) &\longmapsto \big((X, \rho), (X', \rho')\big). \end{aligned} \tag{20}$$

Recall that by definition $\mathcal{N} = \mathcal{N}_0 \times \mathcal{N}_0$. Let $\mathrm{s}_x : \mathcal{N}_0(x) \to \mathcal{N}_0$ be the composition of the morphism st_x with the projection to the first factor $\mathcal{N} \to \mathcal{N}_0$, and let $\mathrm{t}_x : \mathcal{N}_0(x) \to \mathcal{N}_0$ be the composition of the morphism st_x with the projection to the second factor $\mathcal{N} \to \mathcal{N}_0$. Let $\mathrm{st}_x^\# : \mathcal{O}_{\mathcal{N}} \to \mathcal{O}_{\mathcal{N}_0(x)}$, $\mathrm{s}_x^\# : \mathcal{O}_{\mathcal{N}_0} \to \mathcal{O}_{\mathcal{N}_0(x)}$ and $\mathrm{t}_x^\# : \mathcal{O}_{\mathcal{N}_0} \to \mathcal{O}_{\mathcal{N}_0(x)}$ be the corresponding local ring homomorphisms.

Lemma 4.4.1. *Let $x \in \mathbb{B}$ be a nonzero element such that $\nu_p(q(x)) = n$ for some integer $n \geqslant 0$. The morphism st_x is a closed immersion and identifies $\mathcal{N}_0(x)$ as an effective Cartier divisor on $\mathcal{N}$. We* *have the following equality of Cartier divisors on $\mathcal{N}$,*

$$\mathcal{D}_{\mathcal{N}}(x) = \mathcal{N}_0(x).$$

Moreover,

(a) *The functor $\mathcal{N}_0(x)$ is represented by a 2 dimensional regular local ring.*

(b) *There exists two elements $t, t' \in \mathcal{O}_{\mathcal{N}_0(x)}$ such that*

(b1) *There exists an element $t_0 \in \mathcal{O}_{\mathcal{N}_0}$ such that $\mathcal{O}_{\mathcal{N}_0} \simeq W[[t_0]]$ and $\mathrm{s}_x^{\#}(t_0) = t$, $\mathrm{t}_x^{\#}(t_0) = t'$.*

(b2) *There exists an invertible element $\nu \in \mathcal{O}_{\mathcal{N}_0(x)}$ such that*

$$\mathcal{O}_{\mathcal{D}_{\mathcal{N}}(x)} = \mathcal{O}_{\mathcal{N}_0(x)} \simeq \begin{cases} W[[t,t']]/\left(\nu p + (t - t'^{p^n})(t^{p^n} - t')\left(\prod\limits_{\substack{a+b=n\\ a,b\geqslant 1}} (t^{p^{a-1}} - t'^{p^{b-1}})\right)^{p-1}\right), & \text{if } n \geqslant 1; \\ W[[t,t']]/(t-t'), & \text{if } n = 0. \end{cases}$$

Proof. The equality $\mathcal{D}_{\mathcal{N}}(x) = \mathcal{N}_0(x)$ is proved by Zhu [Zhu25, Theorem 6.2.3]. The statement (a) is [KM85, Theorem 5.1.1]. For (b), when $n = 0$, this is the cancellation law. When $n > 0$, note that [KM85, Theorem 13.4.7] states that

$$\mathcal{O}_{\mathcal{N}_0(x)_{\mathbb{F}}} \simeq \mathbb{F}[[t,t']]/\left((t - t'^{p^n})(t^{p^n} - t')\left(\prod_{\substack{a+b=n\\ a,b\geqslant 1}} (t^{p^{a-1}} - t'^{p^{b-1}})\right)^{p-1}\right).$$

Since we also know $\mathcal{O}_{\mathcal{N}_0(x)}$ is a regular local ring, it has to be the form as claimed. □

4.5. **Blow up of the cyclic deformation space.** For an element $x \in \mathbb{B}$ such that $\nu_p(q(x)) \geqslant 0$. Let $\pi_x : \widetilde{\mathcal{N}}_0(x) \to \mathcal{N}_0(x)$ be the blow up morphism of the formal scheme $\mathcal{N}_0(x)$ along its unique closed $\mathbb{F}$-point. We will see that the exceptional divisor of this blow up is isomorphic to $\mathbb{P}^1_{\mathbb{F}}$. For integers k, we use $\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}}}(k)$ to denote the line bundle on $\mathbb{P}^1_{\mathbb{F}}$ of degree k. A formal scheme Z over $\operatorname{Spf} W$ is called horizontal if it is flat over $\operatorname{Spf} W$. Recall that for a line bundle $\mathcal{L}$ on a (formal) scheme X, we use $\mathcal{L}$ to denote element $[\mathcal{O}_X] - [\mathcal{L}]$ in the group $K_0(X)$.

Lemma 4.5.1. *For an element $x \in \mathbb{B}$ such that $n := \nu_p(q(x)) \geqslant 0$. Then*

(a) *The formal scheme $\widetilde{\mathcal{N}}_0(x)$ is a 2-dimensional regular formal scheme and $\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)} \simeq \mathbb{P}^1_{\mathbb{F}}$. We also have an equality $[\mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x)}} \mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}}] = \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}}}(-1)$ in $\mathrm{Gr}^2 K_0^{\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}}(\widetilde{\mathcal{N}}_0(x)) \simeq \mathrm{Gr}^1 K_0(\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}) \simeq \mathrm{Pic}(\mathbb{P}^1_{\mathbb{F}})$.*

(b) *If $n \geqslant 1$, the multiplicity $r(n)$ of the exceptional divisor $\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}$ in the divisor $\mathrm{div}(p) = \widetilde{\mathcal{N}}_0(x)_{\mathbb{F}}$ is given by*

$$r(n) = p^{[n/2]} \cdot \begin{cases} 1 + p^{-1}, & \text{if } n \text{ is even;} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

(c) *Let $\mathcal{C} = \operatorname{Spf} R \subset \mathcal{N}_0(x)$ be a regular horizontal divisor where R is a regular local ring, then*

$$\pi_x^* \mathcal{C} = \widetilde{\mathcal{C}} + \mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)},$$

where $\widetilde{\mathcal{C}} \subset \widetilde{\mathcal{N}}_0(x)$ *is the strict transform of* $\mathcal{C}$ *under the blow up morphism* π_x*, and isomorphic to* $\mathcal{C}$ *under the morphism* π_x*. Moreover,*

$$[\mathcal{O}_{\pi_x^*\mathcal{C}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x)}} \mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}}] = \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}}}(0), \quad [\mathcal{O}_{\widetilde{\mathcal{C}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x)}} \mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}}] = \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}}}(1)$$

as elements in $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{F}})$*.*

(d) *Let* $\mathcal{C}_1 = \mathrm{Spf}\, R_1, \mathcal{C}_2 = \mathrm{Spf}\, R_2 \subset \mathcal{N}_0(x)$ *be two regular horizontal divisors such that* R_1 *and* R_2 *are two regular local rings and* $\mathcal{C}_1 \neq \mathcal{C}_2$*, then*

$$\chi(\widetilde{\mathcal{N}}_0(x), \mathcal{O}_{\pi_x^*\mathcal{C}_1} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x)}} \mathcal{O}_{\pi_x^*\mathcal{C}_2}) = \chi(\mathcal{N}_0(x), \mathcal{O}_{\mathcal{C}_1} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(x)}} \mathcal{O}_{\mathcal{C}_2}),$$

Proof. We only give the proof for $n \geqslant 1$ (the case $n = 0$ is similar and easier). We first prove (a) and (b). By Lemma 4.4.1 (b2), the formal scheme $\widetilde{\mathcal{N}}_0(x)$ is covered by the following two open formal subschemes:

$$U_0 = \mathrm{Spf}\, W[u][[t]]/\left(\nu p + t^{r(n)} \cdot f_1(u,t)\right), \quad U_1 = \mathrm{Spf}\, W[v][[t']]/\left(\nu p + t'^{r(n)} \cdot f_2(v,t')\right), \tag{21}$$

where $uv = 1$ and $f_1(u,t) \in W[u][[t]], f_2(v,t') \in W[v][[t']]$ are two non-unit elements such that $f_1(u,0) \neq 0$ and $f_2(v,0) \neq 0$. Therefore (a) and (b) are true by the explicit description of $\widetilde{\mathcal{N}}_0(x)$ in (21).

Now we prove (c). We write $\mathcal{O}_{\mathcal{N}_0(x)} \simeq W[[t,t']]/(d_x)$ as in Lemma 4.4.1. By the regularity of the divisor $\mathcal{C}$, the equation $f_{\mathcal{C}} \in \mathcal{O}_{\mathcal{N}_0(x)} \simeq W[[t,t']]/(d_x)$ cutting out the divisor $\mathcal{C}$ must take the following form

$$f_{\mathcal{C}} \equiv a \cdot t + a' \cdot t' \bmod (p,t,t')^2,$$

where $a, a' \in W$ and at least one of them is a unit. Then under the explicit equation (21) of $\widetilde{\mathcal{N}}_0(x)$, the equation $f_{\pi^*\mathcal{C}}$ of the pullback $\pi_x^*\mathcal{C}$ takes the following form:

$$f_{\pi^*\mathcal{C}}|_{U_0} = t \cdot (a + a'u) \bmod t^2, \quad f_{\pi^*\mathcal{C}}|_{U_1} = t' \cdot (av + a') \bmod t'^2.$$

Therefore we have $\pi^*\mathcal{C} = \mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)} + \mathcal{C}_1$ where $\mathcal{C}_1$ is an irreducible divisor on $\widetilde{\mathcal{N}}_0(x)$. The scheme $\mathcal{C}_1 \cap \mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}$ a single point scheme-theoretically since at least one of a, a' is a unit. Therefore

$$[\mathcal{O}_{\widetilde{\mathcal{C}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x)}} \mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}}] = \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}}}(1).$$

Notice that the maximal ideal of $\mathcal{C}$ is already principal since $\mathcal{C}$ is a 1-dimensional regular local ring, therefore the strict transform of $\widetilde{\mathcal{C}}$ is isomorphic to $\mathcal{C}$ under the morphism π_x. Notice that we have $\widetilde{\mathcal{C}} \subset \mathcal{C}_1$, hence $\widetilde{\mathcal{C}} = \mathcal{C}_1$ by the irreducibility of both divisors. Therefore

$$[\mathcal{O}_{\pi_x^*\mathcal{C}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x)}} \mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}}] = [\mathcal{O}_{\mathcal{C}_1} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x)}} \mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}}] + [\mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x)}} \mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}}] = \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}}}(0).$$

Now we prove (d). Using the intersection pairing on the divisors on a regular surface, we have

$$\begin{aligned}\chi(\widetilde{\mathcal{N}}_0(x), \mathcal{O}_{\pi_x^*\mathcal{C}_1} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x)}} \mathcal{O}_{\pi_x^*\mathcal{C}_2}) &= (\pi_x^*\mathcal{C}_1 \cdot \pi_x^*\mathcal{C}_2)_{\widetilde{\mathcal{N}}_0(x)} = \left((\widetilde{\mathcal{C}}_1 + \mathrm{Exc}_{\widetilde{\mathcal{N}}_0(x)}) \cdot \pi_x^*\mathcal{C}_2\right)_{\widetilde{\mathcal{N}}_0(x)} \\ &= \left(\widetilde{\mathcal{C}}_1 \cdot \pi_x^*\mathcal{C}_2\right)_{\widetilde{\mathcal{N}}_0(x)} \overset{\text{projection formula}}{=} (\mathcal{C}_1 \cdot \mathcal{C}_2)_{\mathcal{N}_0(x)} \\ &= \chi(\mathcal{N}_0(x), \mathcal{O}_{\mathcal{C}_1} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(x)}} \mathcal{O}_{\mathcal{C}_2}).\end{aligned}$$

□

4.6. **Rapoport–Zink space with level $\Gamma_0(p)$.** Let $x_0 \in \mathbb{B}$ be an element such that $\nu_p(q(x_0)) = 1$. The cyclic deformation space $\mathcal{N}_0(x_0)$ is a Rapoport–Zink space with level $\Gamma_0(p)$. Let

$$\left((X^{\mathrm{univ}} \xrightarrow{x_0^{\mathrm{univ}}} X'^{\mathrm{univ}}), (\rho^{\mathrm{univ}}, \rho'^{\mathrm{univ}})\right)$$

be the universal deformation of the quasi-isogeny x_0 over the formal scheme $\mathcal{N}_0(x_0)$.

Let $\mathfrak{m} \subset \mathcal{O}_{\mathcal{N}_0(x_0)}$ be the maximal ideal of the local ring $\mathcal{O}_{\mathcal{N}_0(x_0)}$. Let $\mathcal{N}_0(x_0)_{\mathbb{F}} := \mathcal{N}_0(x_0) \times_W \mathbb{F}$ be the reduction mod p of the formal scheme $\mathcal{N}_0(x_0)$, it is a formal scheme over $\mathbb{F}$. The following lemma is clear from the above description of the local ring $\mathcal{O}_{\mathcal{N}_0(x_0)}$.

Lemma 4.6.1. *The following facts hold:*

(a) *The element $p \in \mathfrak{m}^2 \backslash \mathfrak{m}^3$.*

(b) *By Lemma 4.4.1 (b), there exists two elements $t, t' \in \mathcal{O}_{\mathcal{N}_0(x_0)}$ such that*

$$\mathcal{O}_{\mathcal{N}_0(x_0)} \simeq W[[t,t']]/\left(\nu p + (t^p - t')(t - t'^p)\right). \tag{22}$$

The formal scheme $\mathcal{N}_0(x_0)_{\mathbb{F}}$ has two irreducible components $\mathcal{N}_0(x_0)^{\mathrm{F}}$ and $\mathcal{N}_0(x_0)^{\mathrm{V}}$ such that:

- *The two irreducible components $\mathcal{N}_0(x_0)^{\mathrm{F}}$ and $\mathcal{N}_0(x_0)^{\mathrm{V}}$ intersect properly at the unique closed point of $\mathcal{N}_0(x_0)_{\mathbb{F}}$.*
- *The formal scheme $\mathcal{N}_0(x_0)^{\mathrm{F}} \simeq \mathrm{Spf}\, \mathbb{F}[[t]]$. Over the formal scheme $\mathcal{N}_0(x_0)^{\mathrm{F}}$, the universal isogeny $X^{\mathrm{univ}} \xrightarrow{x_0^{\mathrm{univ}}} X'^{\mathrm{univ}}$ is isomorphic to the Frobenius morphism.*
- *The formal scheme $\mathcal{N}_0(x_0)^{\mathrm{V}} \simeq \mathrm{Spf}\, \mathbb{F}[[t']]$. Over the formal scheme $\mathcal{N}_0(x_0)^{\mathrm{V}}$, the universal isogeny $X^{\mathrm{univ}} \xrightarrow{x_0^{\mathrm{univ}}} X'^{\mathrm{univ}}$ is isomorphic to the Verschiebung morphism.*

Proof. The part (a) is clear by the explicit equation (22). For part (b), let $\mathcal{N}_0(x_0)^{\mathrm{F}}$ (resp. $\mathcal{N}_0(x_0)^{\mathrm{V}}$) be the closed formal subscheme cut out by the equation $t' - t^p$ (resp. $t - t'^p$), therefore we have $\mathcal{N}_0(x_0)^{\mathrm{F}} \simeq \mathrm{Spf}\, \mathbb{F}[[t]]$ (resp. $\mathcal{N}_0(x_0)^{\mathrm{V}} \simeq \mathrm{Spf}\, \mathbb{F}[[t']]$). It's easy to see that the two irreducible components $\mathcal{N}_0(x_0)^{\mathrm{F}}$ and $\mathcal{N}_0(x_0)^{\mathrm{V}}$ intersect properly at the unique closed point of $\mathcal{N}_0(x_0)_{\mathbb{F}}$.

By the equation $t' = t^p$, we have

$$X'^{\mathrm{univ}} \simeq \left(X^{\mathrm{univ}}\right)^{(p)}.$$

Therefore the universal isogeny $X^{\mathrm{univ}} \xrightarrow{x_0^{\mathrm{univ}}} X'^{\mathrm{univ}}$ is isomorphic to the Frobenius morphism over $\mathcal{N}_0(x_0)^{\mathrm{F}}$. The statement for $\mathcal{N}_0(x_0)^{\mathrm{V}}$ can be proved in the same way. We refer the readers to [KM85, §13.4] for more details. □

Proposition 4.6.2. *There exists a basis e, f for the rank 2 free $\mathcal{O}_{\mathcal{N}_0(x_0)}$-module $\mathbb{D}(X^{\mathrm{univ}})_{\mathcal{N}_0(x_0)}$ and a basis e', f' for the rank 2 free $\mathcal{O}_{\mathcal{N}_0(x_0)}$-module $\mathbb{D}(X'^{\mathrm{univ}})_{\mathcal{N}_0(x_0)}$ such that the Hodge filtrations are given by*

$$0 \to \mathrm{Fil}^1\mathbb{D}(X^{\mathrm{univ}})_{\mathcal{N}_0(x_0)} = \mathcal{O}_{\mathcal{N}_0(x_0)} \cdot (f + xe) \to \mathbb{D}(X^{\mathrm{univ}})_{\mathcal{N}_0(x_0)}, \tag{23}$$

$$0 \to \mathrm{Fil}^1\mathbb{D}(X'^{\mathrm{univ}})_{\mathcal{N}_0(x_0)} = \mathcal{O}_{\mathcal{N}_0(x_0)} \cdot (f' + ye') \to \mathbb{D}(X'^{\mathrm{univ}})_{\mathcal{N}_0(x_0)}, \tag{24}$$

and the morphism $\mathbb{D}(x_0)_{\mathcal{N}_0(x_0)}$ takes the following form

$$\mathbb{D}(x_0)_{\mathcal{N}_0(x_0)}[e,f] = [e',f']\begin{pmatrix} 0 & -p \\ 1 & 0 \end{pmatrix}.$$

Moreover, the local ring $\mathcal{O}_{\mathcal{N}_0(x_0)}$ *is isomorphic to* $W[[x,y]]/(p+xy)$.

Proof. Scaling the basis $[e_1, e_2]$ of the rank 2 free W-module $\mathbb{D}(\mathbb{X})$ in the end of §4.1 by invertible elements in W, we can assume that the morphism $\mathbb{D}(x_0)_W$ takes the form $\mathbb{D}(x_0)_W[e_1,e_2] = [e_1,e_2]\begin{pmatrix} 0 & -p \\ 1 & 0 \end{pmatrix}$.

As $\mathcal{O}_{\mathcal{N}_0(x_0)}$-modules, we have $\mathbb{D}(X^{\mathrm{univ}})_{\mathcal{N}_0(x_0)} \simeq \mathbb{D}(\mathbb{X})\otimes_W \mathcal{O}_{\mathcal{N}_0(x_0)}$ and $\mathbb{D}(X'^{\mathrm{univ}})_{\mathcal{N}_0(x_0)} \simeq \mathbb{D}(\mathbb{X})\otimes_W \mathcal{O}_{\mathcal{N}_0(x_0)}$. By the closed immersion $\operatorname{Spec}\mathbb{F} \to \mathcal{N}_0(x_0)$ and the property of crystals, the morphism $\mathbb{D}(x_0)_{\mathcal{N}_0(x_0)}$ takes the following form after scaling the elements e_1 and e_2 by invertible elements in the local ring $\mathcal{O}_{\mathcal{N}_0(x_0)}$:

$$\mathbb{D}(x_0)_{\mathcal{N}_0(x_0)}[e_1,e_2] = [e_1', e_2']\begin{pmatrix} h & -p+mh \\ 1 & m \end{pmatrix}$$

for some elements $h, m \in \mathfrak{m}$. Let $e = e_1,\ f = e_2 - me_1,\ e' = e_1',\ f' = e_2' + he_1'$. We have

$$\mathbb{D}(x_0)_{\mathcal{N}_0(x_0)}[e,f] = [e', f']\begin{pmatrix} 0 & -p \\ 1 & 0 \end{pmatrix}. \tag{25}$$

Recall that $\mathcal{O}_{\mathcal{N}_0} \simeq W[[t]]$. Therefore the local ring $\mathcal{O}_{\mathcal{N}}$ is isomorphic to $W[[t,t']]$. We can choose two uniformizers t and t' such that the Hodge filtrations on $\mathbb{D}(X^{\mathrm{univ}})_{\mathcal{N}}$ and $\mathbb{D}(X'^{\mathrm{univ}})_{\mathcal{N}}$ are given by

$$0 \to \mathrm{Fil}^1\mathbb{D}(X^{\mathrm{univ}})_{\mathcal{N}} = \mathcal{O}_{\mathcal{N}}\cdot(e_2 + te_1) \to \mathbb{D}(X^{\mathrm{univ}})_{\mathcal{N}}.$$
$$0 \to \mathrm{Fil}^1\mathbb{D}(X'^{\mathrm{univ}})_{\mathcal{N}} = \mathcal{O}_{\mathcal{N}}\cdot(e_2' + t'e_1') \to \mathbb{D}(X^{\mathrm{univ}})_{\mathcal{N}}.$$

Then the Hodge filtrations on $\mathbb{D}(X^{\mathrm{univ}})_{\mathcal{N}_0(x_0)}$ and $\mathbb{D}(X'^{\mathrm{univ}})_{\mathcal{N}_0(x_0)}$ are given by

$$0 \to \mathcal{O}_{\mathcal{N}_0(x_0)}\cdot(e_2 + te_1) = \mathcal{O}_{\mathcal{N}_0(x_0)}\cdot(f + (t+m)e) \to \mathbb{D}(X^{\mathrm{univ}})_{\mathcal{N}_0(x_0)}.$$
$$0 \to \mathcal{O}_{\mathcal{N}_0(x_0)}\cdot(e_2' + t'e_1') = \mathcal{O}_{\mathcal{N}_0(x_0)}\cdot\big(f' + (t'-h)e'\big) \to \mathbb{D}(X^{\mathrm{univ}})_{\mathcal{N}_0(x_0)}.$$

Let $x = t+m \in \mathfrak{m}, y = t'-h \in \mathfrak{m}$. Since x_0 lifts to an isogeny over the formal scheme $\mathcal{N}_0(x_0)$, we have $\mathbb{D}(x_0)_{\mathcal{N}_0(x_0)}(f+xe) \subset \mathcal{O}_{\mathcal{N}_0(x_0)}\cdot(f'+ye')$. By (25), we have $\mathbb{D}(x_0)_{\mathcal{N}_0(x_0)}(f+xe) = xf' - pe' \in \mathcal{O}_{\mathcal{N}_0(x_0)}\cdot(f'+ye')$. Therefore $p + xy = 0$.

Next we want to show that $\mathfrak{m} = (x,y)$. By the equation $p+xy = 0$, the two irreducible components of the formal scheme $\mathcal{N}_0(x_0)_{\mathbb{F}}$ are given by $x = 0$ and $y = 0$. By Lemma 4.6.1, they intersect properly at the unique closed point of $\mathcal{N}_0(x_0)$, hence $\mathfrak{m} = (x,y)$. Therefore we conclude that $\mathcal{O}_{\mathcal{N}_0(x_0)} \simeq W[[x,y]]/(p+xy)$. □

Remark 4.6.3. We shall make the convention that under the isomorphism $\mathcal{O}_{\mathcal{N}_0(x_0)} \simeq W[[x,y]]/(p+xy)$, the equation $y = 0$ is the equation of the irreducible component $\mathcal{N}_0(x_0)^{\mathrm{F}}$, while the equation $x = 0$ is the equation of the irreducible component $\mathcal{N}_0(x_0)^{\mathrm{V}}$.

Lemma 4.6.4. *Let* $t, t' \in \mathcal{O}_{\mathcal{N}_0(x_0)}$ *be two elements guaranteed by Lemma 4.4.1 (b) for the element* x_0. *Then there exists invertible elements* $\nu_1, \nu_2, \omega_2, \omega_2 \in W[[x,y]]$ *such that*

$$t = \nu_1 x + \nu_2 y^p + p\cdot h,\quad t' = \omega_1 y + \omega_2 x^p + p\cdot h',$$

where h *and* h' *are two elements in the local ring* $\mathcal{O}_{\mathcal{N}_0(x_0)}$.

Proof. Under the coordinate in Proposition 4.6.2, the quotient ring $\mathcal{O}_{\mathcal{N}_0(x_0)}/(p) \simeq \mathbb{F}[[x,y]]/(x\cdot y)$. While under the coordinate t, t', we have $\mathcal{O}_{\mathcal{N}_0(x_0)}/(p) \simeq \mathbb{F}[[t,t']]/\left((t^p - t')(t - t'^p)\right)$. Therefore there exists invertible elements $\nu, \omega \in \mathcal{O}^{\times}_{\mathcal{N}_0(x_0)}$ such that

$$x = \nu \cdot (t - t'^p),\ y = \omega \cdot (t^p - t') \text{ in } \mathcal{O}_{\mathcal{N}_0(x_0)}/(p).$$

The claim in the lemma follows by solving the above equations for t, t'. □

4.7. **The product space $\mathcal{N}_0(x_0) \times_W \mathcal{N}_0(x_0)$.** The formal scheme $\mathcal{N}(x_0) := \mathcal{N}_0(x_0) \times_W \mathcal{N}_0(x_0)$ is represented by the local ring $W[[x_1, y_1, x_2, y_2]]/(p + x_1y_1, p + x_2y_2)$ by Lemma 4.6.2. The sequence $p + x_1y_1, p + x_2y_2 = x_2y_2 - x_1y_1$ is not a regular sequence since the second element is not regular. Hence the local ring is not a regular local ring. Therefore the formal scheme $\mathcal{N}(x_0)$ is not regular.

Notice that $\mathcal{N}(x_0) \times_W \mathbb{F} \simeq \mathcal{N}_0(x_0)_{\mathbb{F}} \times_{\mathbb{F}} \mathcal{N}_0(x_0)_{\mathbb{F}}$. The formal scheme $\mathcal{N}(x_0)_{\mathbb{F}}$ has the following 4 irreducible components by Lemma 4.6.1.

$$\begin{aligned} \mathcal{N}(x_0)^{\mathrm{FF}} &= \mathcal{N}_0(x_0)^{\mathrm{F}} \times_{\mathbb{F}} \mathcal{N}_0(x_0)^{\mathrm{F}},\ \mathcal{N}(x_0)^{\mathrm{FV}} = \mathcal{N}_0(x_0)^{\mathrm{F}} \times_{\mathbb{F}} \mathcal{N}_0(x_0)^{\mathrm{V}}, \\ \mathcal{N}(x_0)^{\mathrm{VV}} &= \mathcal{N}_0(x_0)^{\mathrm{V}} \times_{\mathbb{F}} \mathcal{N}_0(x_0)^{\mathrm{V}},\ \mathcal{N}(x_0)^{\mathrm{VF}} = \mathcal{N}_0(x_0)^{\mathrm{V}} \times_{\mathbb{F}} \mathcal{N}_0(x_0)^{\mathrm{F}}. \end{aligned} \tag{26}$$

Let $\widehat{\mathbb{A}}^2_{\mathbb{F}}$ be the completion of the 2-dimensional affine plane at the point $(0,0)$. By Lemma 4.6.1 (b), the formal schemes $\mathcal{N}(x_0)^{\mathrm{FF}}, \mathcal{N}(x_0)^{\mathrm{VV}}, \mathcal{N}(x_0)^{\mathrm{FV}}, \mathcal{N}(x_0)^{\mathrm{VF}}$ are all isomorphic to $\widehat{\mathbb{A}}^2_{\mathbb{F}}$.

We define two morphisms $s_+, s_- : \mathcal{N}(x_0) \to \mathcal{N}$ by the moduli interpretations: let S be a scheme in the category Nilp_W,

$$s_+ : \left(\left(X_1 \xrightarrow{\pi_1} X_1', (\rho_1, \rho_1')\right), \left(X_2 \xrightarrow{\pi_2} X_2', (\rho_2, \rho_2')\right)\right) \in \mathcal{N}(x_0)(S) \mapsto ((X_1, \rho_1), (X_2, \rho_2)) \in \mathcal{N}(S); \tag{27}$$

$$s_- : \left(\left(X_1 \xrightarrow{\pi_1} X_1', (\rho_1, \rho_1')\right), \left(X_2 \xrightarrow{\pi_2} X_2', (\rho_2, \rho_2')\right)\right) \in \mathcal{N}(x_0)(S) \mapsto \left((X_1', \rho_1'), (X_2', \rho_2')\right) \in \mathcal{N}(S). \tag{28}$$

4.8. **Local Hecke correspondences.** Let $x_0 \in \mathbb{B}$ be an element such that $\nu_p(q(x_0)) = 1$. Let $(\cdot)' : \mathbb{B} \to \mathbb{B},\ b \mapsto b' := x_0 \cdot b \cdot x_0^{-1}$ be the conjugation-by-x_0 automorphism on $\mathbb{B}$. Let x be an element in $\mathbb{B}$ such that $n = \nu_p(q(x)) \geqslant 1$. For an object $(X \xrightarrow{\pi} X', (\rho, \rho')) \in \mathcal{N}_0(x)(S)$, there is a standard decomposition of the cyclic isogeny π into n degree p isogenies:

$$\pi : X = X_0 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_{n-1}} X_{n-1} \xrightarrow{\pi_n} X_n = X'. \tag{29}$$

We refer to [KM85, §6.7] for the details on the notions of standard decomposition.

Let S be a W-scheme such that p is locally nilpotent on S, for any object $(X \xrightarrow{\pi} X', (\rho, \rho')) \in \mathcal{N}_0(x)(S)$ with the standard decomposition of π as (29). We construct two height 0 quasi-isogenies $\rho_1 : \mathbb{X} \times_{\mathbb{F}} \overline{S} \to X_1 \times_S \overline{S}$ and $\rho_{n-1} : \mathbb{X} \times_{\mathbb{F}} \overline{S} \to X_{n-1} \times_S \overline{S}$ in the following way,

$$\rho_1 = (\pi_1 \times_S \overline{S}) \circ \rho \circ (x_0 \times_{\mathbb{F}} \overline{S})^{-1},\ \rho_{n-1} = (\pi_n \times_S \overline{S})^{-1} \circ \rho \circ (x_0 \times_{\mathbb{F}} \overline{S}).$$

$$\rho_1^- = (\pi_1^\vee \times_S \overline{S})^{-1} \circ \rho \circ (x_0 \times_{\mathbb{F}} \overline{S}),\ \rho_{n-1}^+ = (\pi_n^\vee \times_S \overline{S}) \circ \rho \circ (x_0 \times_{\mathbb{F}} \overline{S})^{-1}.$$

Then we have

$$\left(X \xrightarrow{\pi_1} X_1, (\rho, \rho_1)\right), \left(X_{n-1} \xrightarrow{\pi_n} X', (\rho_{n-1}, \rho')\right), \left(X_1 \xrightarrow{\pi_1^\vee} X, (\rho_1^-, \rho)\right), \left(X' \xrightarrow{\pi_n^\vee} X_{n-1}, (\rho', \rho_{n-1}^+)\right) \in \mathcal{N}_0(x_0)(S).$$

Define four morphisms $\mathrm{st}_x^{\mathrm{I}+}, \mathrm{st}_x^{\mathrm{I}-}, \mathrm{st}_x^{\mathrm{II}+}, \mathrm{st}_x^{\mathrm{II}-}$ as follows,

$$\mathrm{st}_x^{\mathrm{I}+}\colon\ (X \xrightarrow{\pi} X', (\rho,\rho')) \in \mathcal{N}_0(x)(S) \mapsto \left(\left(X \xrightarrow{\pi_1} X_1, (\rho,\rho_1)\right), \left(X_{n-1} \xrightarrow{\pi_n} X', (\rho_{n-1},\rho')\right)\right) \in \mathcal{N}(x_0)(S),$$
$$\mathrm{st}_x^{\mathrm{I}-}\colon\ (X \xrightarrow{\pi} X', (\rho,\rho')) \in \mathcal{N}_0(x)(S) \mapsto \left(\left(X_{n-1} \xrightarrow{\pi_n} X', (\rho_{n-1},\rho')\right), \left(X \xrightarrow{\pi_1} X_1, (\rho,\rho_1)\right)\right) \in \mathcal{N}(x_0)(S),$$
$$\mathrm{st}_x^{\mathrm{II}+}\colon\ (X \xrightarrow{\pi} X', (\rho,\rho')) \in \mathcal{N}_0(x)(S) \mapsto \left(\left(X \xrightarrow{\pi_1} X_1, (\rho,\rho_1)\right), \left(X' \xrightarrow{\pi_n^\vee} X_{n-1}, (\rho',\rho_{n-1}^+)\right)\right) \in \mathcal{N}(x_0)(S),$$
$$\mathrm{st}_x^{\mathrm{II}-}\colon\ (X \xrightarrow{\pi} X', (\rho,\rho')) \in \mathcal{N}_0(x)(S) \mapsto \left(\left(X_1 \xrightarrow{\pi_1^\vee} X, (\rho_1^-,\rho)\right), \left(X_{n-1} \xrightarrow{\pi_n} X', (\rho_{n-1},\rho')\right)\right) \in \mathcal{N}(x_0)(S).$$

Lemma 4.8.1. *Let x be an element in $\mathbb{B}$ such that $n = \nu_p(q(x)) \geqslant 1$. Then the four morphisms $\mathrm{st}_x^{\mathrm{I}+}, \mathrm{st}_x^{\mathrm{I}-}, \mathrm{st}_x^{\mathrm{II}+}, \mathrm{st}_x^{\mathrm{II}-} : \mathcal{N}_0(x) \to \mathcal{N}(x_0)$ are closed immersions.*

Proof. We only give the proof for the morphism $\mathrm{st}_x^{\mathrm{I}+}$. Notice that

$$(s_{x_0} \times t_{x_0}) \circ \mathrm{st}_x^{\mathrm{I}+} = \mathrm{st}_x : \mathcal{N}_0(x) \to \mathcal{N}.$$

The morphism st_x is a closed immersion by Lemma 4.4.1, hence $\mathrm{st}_x^{\mathrm{I}+}$ is a closed immersion. □

Lemma 4.8.2. *Let x be an element in $\mathbb{B}$ such that $n = \nu_p(q(x)) \geqslant 1$. Then the following diagram is Cartesian,*

$$\begin{array}{ccc} \operatorname{Spec}\mathbb{F} & \longrightarrow & \mathcal{N}(x_0)^{\mathrm{VF}} \\ \big\downarrow & & \big\downarrow \\ \mathcal{N}_0(x) & \xrightarrow{\mathrm{st}_x^{\mathrm{I}+}} & \mathcal{N}(x_0). \end{array}$$

Proof. It is proved in [KM85, Theorem 13.3.5] that a cyclic isogeny π of degree p^n between two p-divisible groups over an $\mathbb{F}$-scheme S whose first decomposition π_1 is isomorphic to Verschiebung morphism V is isomorphic to V^n. Then by Lemma 4.4.1 (b2), the fiber product $\mathcal{N}_0(x) \times_{\mathrm{st}_x^{\mathrm{I}+}, \mathcal{N}(x_0)} \mathcal{N}(x_0)^{\mathrm{VF}}$ is given by the following two equations

$$t'^{p^n} - t \quad \text{and} \quad t' - t^{p^n}.$$

These two elements generate the maximal ideal of the unique closed $\mathbb{F}$-point of $\mathcal{N}_0(x)$, hence $\mathcal{N}_0(x) \times_{\mathrm{st}_x^{\mathrm{I}+}, \mathcal{N}(x_0)} \mathcal{N}(x_0)^{\mathrm{VF}} \simeq \operatorname{Spec}\mathbb{F}$. □

Remark 4.8.3. We can summarize the above lemma as $\mathcal{N}_0(x) \times_{\mathrm{st}_x^{\mathrm{I}+}, \mathcal{N}(x_0)} \mathcal{N}(x_0)^{\mathrm{VF}} \simeq \operatorname{Spec}\mathbb{F}$. Similar arguments also imply that $\mathcal{N}_0(x) \times_{\mathrm{st}_x^{\mathrm{I}-}, \mathcal{N}(x_0)} \mathcal{N}(x_0)^{\mathrm{FV}} \simeq \operatorname{Spec}\mathbb{F}$, $\mathcal{N}_0(x) \times_{\mathrm{st}_x^{\mathrm{II}+}, \mathcal{N}(x_0)} \mathcal{N}(x_0)^{\mathrm{VV}} \simeq \operatorname{Spec}\mathbb{F}$ and $\mathcal{N}_0(x) \times_{\mathrm{st}_x^{\mathrm{II}-}, \mathcal{N}(x_0)} \mathcal{N}(x_0)^{\mathrm{FF}} \simeq \operatorname{Spec}\mathbb{F}$.

4.9. **Special cycles on the product: the $\Gamma_0(p) \times_{\mathbb{G}_m} \Gamma_0(p)$ case.** The formal scheme $\mathcal{N}(x_0)$ parameterizes a pair of deformations of the quasi-isogeny x_0. Let

$$\left(X_1^{\mathrm{univ}} \xrightarrow{x_{0,1}^{\mathrm{univ}}} X_1'^{\mathrm{univ}}, (\rho_1^{\mathrm{univ}}, \rho_1'^{\mathrm{univ}})\right), \quad \left(X_2^{\mathrm{univ}} \xrightarrow{x_{0,2}^{\mathrm{univ}}} X_2'^{\mathrm{univ}}, (\rho_2^{\mathrm{univ}}, \rho_2'^{\mathrm{univ}})\right) \tag{30}$$

be the universal pairs over the formal scheme $\mathcal{N}(x_0)$.

Recall our convention that $x' = x_0 \cdot x \cdot x_0^{-1}$. Therefore we have the following commutative diagram

$$\begin{array}{ccc} X_1^{\mathrm{univ}} & \overset{x}{\dashrightarrow} & X_2^{\mathrm{univ}} \\ {\scriptstyle x_{0,1}^{\mathrm{univ}}}\big\downarrow & & \big\downarrow{\scriptstyle x_{0,2}^{\mathrm{univ}}} \\ X_1'^{\mathrm{univ}} & \overset{x'}{\dashrightarrow} & X_2'^{\mathrm{univ}}, \end{array}$$

here the dotted arrows below x and x' mean that they are quasi-isogenies.

Definition 4.9.1. For any subset $H \subset \mathbb{B}$. Define the special cycle $\mathcal{Z}_{\mathcal{N}(x_0)}(H) \subset \mathcal{N}(x_0)$ to be the closed formal subscheme cut out by the conditions,

$$\rho_2^{\mathrm{univ}} \circ x \circ (\rho_1^{\mathrm{univ}})^{-1} \in \mathrm{Hom}(X_1^{\mathrm{univ}}, X_2^{\mathrm{univ}});$$
$$\rho_2'^{\mathrm{univ}} \circ x' \circ (\rho_1'^{\mathrm{univ}})^{-1} \in \mathrm{Hom}(X_1'^{\mathrm{univ}}, X_2'^{\mathrm{univ}}).$$

for all $x \in H$.

Define the special cycle $\mathcal{Y}_{\mathcal{N}(x_0)}(H) \subset \mathcal{N}(x_0)$ to be the closed formal subscheme cut out by the conditions,

$$\rho_2'^{\mathrm{univ}} \circ x_0 \cdot x \circ (\rho_1^{\mathrm{univ}})^{-1} \in \mathrm{Hom}(X_1^{\mathrm{univ}}, X_2^{\mathrm{univ}});$$
$$\rho_2^{\mathrm{univ}} \circ \overline{x_0} \cdot x' \circ (\rho_1'^{\mathrm{univ}})^{-1} \in \mathrm{Hom}(X_1'^{\mathrm{univ}}, X_2'^{\mathrm{univ}}).$$

for all $x \in H$.

Lemma 4.9.2. *Let $x \in \mathbb{B}$ be an element such that $\nu_p(q(x)) \geqslant 0$. The morphisms $\mathrm{st}_{x_0\cdot x}^{\mathrm{I}+}$, $\mathrm{st}_{x_0\cdot\overline{x}}^{\mathrm{I}-}$, $\mathrm{st}_{x}^{\mathrm{II}+}$ and $\mathrm{st}_{x'}^{\mathrm{II}-}$ (the later two morphisms are only defined for elements x such that $\nu_p(q(x)) \geqslant 1$) sends the corresponding source formal scheme to the special divisor $\mathcal{Z}_{\mathcal{N}(x_0)}(x)$.*

Proof. Let's first consider the morphism $\mathrm{st}_{x_0\cdot x}^{\mathrm{I}+} : \mathcal{N}_0(x_0 \cdot x) \to \mathcal{N}(x_0)$. We still use $\mathcal{N}_0(x_0 \cdot x)$ to denote the image in $\mathcal{N}(x_0)$ of the closed immersion $\mathrm{st}_{x_0\cdot x}^{\mathrm{I}+}$. Denote by

$$\left(X_1^{\mathrm{I}+} \xrightarrow{(x_0)_1^{\mathrm{I}+}} X_1'^{\mathrm{I}+}, \left(\rho_1^{\mathrm{I}+}, \rho_1'^{\mathrm{I}+}\right)\right), \quad \left(X_2^{\mathrm{I}+} \xrightarrow{(x_0)_2^{\mathrm{I}+}} X_2'^{\mathrm{I}+}, \left(\rho_2^{\mathrm{I}+}, \rho_2'^{\mathrm{I}+}\right)\right)$$

be the base change of the universal pair (30) to the formal scheme $\mathcal{N}_0(x_0 \cdot x)$ through the morphism $\mathrm{st}_{x_0\cdot x}^{\mathrm{I}+}$.

The quasi-isogeny $x_0 \cdot x$ lifts to a cyclic isogeny $(x_0 \cdot x)^{\mathrm{I}+} : X_1^{\mathrm{I}+} \to X_2'^{\mathrm{I}+}$. By the definition of the morphism $\mathrm{st}_{x_0\cdot x}^{\mathrm{I}+}$ in 4.8, the isogeny $\left(X_2^{\mathrm{I}+} \xrightarrow{(x_0)_2^{\mathrm{I}+}} X_2'^{\mathrm{I}+}\right)$ is the last term in the factorization of the cyclic isogeny $(x_0 \cdot x)^{\mathrm{I}+}$. Therefore the quasi-isogeny $\left((x_0)_2^{\mathrm{I}+}\right)^{-1} \circ (x_0 \cdot x)^{\mathrm{I}+} : X_1^{\mathrm{I}+} \to X_2^{\mathrm{I}+}$ is an isogeny. Notice that

$$\left(X_1^{\mathrm{I}+} \xrightarrow{\left((x_0)_2^{\mathrm{I}+}\right)^{-1} \circ (x_0\cdot x)^{\mathrm{I}+}} X_2^{\mathrm{I}+}, (\rho_1^{\mathrm{I}+}, \rho_2^{\mathrm{I}+})\right)$$

is a lift of the quasi-isogeny x. Therefore the morphism $\mathrm{st}_{x_0\cdot x}^{\mathrm{I}+}$ maps $\mathcal{N}_0(x_0 \cdot x)$ to $\mathcal{Z}_{\mathcal{N}(x_0)}(x)$.

The ideas and proofs for other morphisms $\mathrm{st}_{x_0\cdot\overline{x}}^{\mathrm{I}-}$, $\mathrm{st}_{x}^{\mathrm{II}+}$ and $\mathrm{st}_{x'}^{\mathrm{II}-}$ are similar. □

Lemma 4.9.3. *Let $x \in \mathbb{B}$ be an element.*

(a) *If $\nu_p(q(x)) = 0$, we have the following equality of closed formal subschemes of $\mathcal{N}(x_0)$:*

$$\mathrm{st}_{x_0\cdot x}^{\mathrm{I}+}\left(\mathcal{N}_0(x_0 \cdot x)\right) = \mathrm{st}_{x_0\cdot\overline{x}}^{\mathrm{I}-}\left(\mathcal{N}_0(x_0 \cdot \overline{x})\right).$$

(b) *If $\nu_p(q(x)) = 1$, we have the following equality of closed formal subschemes of $\mathcal{N}(x_0)$:*

$$\mathrm{st}_{x}^{\mathrm{II}+}\left(\mathcal{N}_0(x)\right) = \mathrm{st}_{x'}^{\mathrm{II}-}\left(\mathcal{N}_0(x')\right).$$

Proof. Let S be a W-scheme such that p is locally nilpotent on S. We first prove (a). Let

$$\left(X_1 \xrightarrow{\pi_1} X_1', (\rho_1, \rho_1')\right), \left(X_2 \xrightarrow{\pi_2} X_2', (\rho_2, \rho_2')\right) \tag{31}$$

be an object in the set $\mathrm{st}^{\mathrm{I}+}_{x_0\cdot x}\left(\mathcal{N}_0(x_0\cdot x)\right)(S)$. By the definition of the morphism $\mathrm{st}^{\mathrm{I}+}_{x_0\cdot x}$ there exist two isomorphisms $\phi : X_1 \to X_2$ and $\phi' : X_1' \to X_2'$ such that $\pi_2 \circ \phi = \phi' \circ \pi_1$. Then it's easy to see that the object

$$\left(X_2 \xrightarrow{\pi_1\circ\phi^\vee} X_1', (\rho_2, \rho_1')\right) \in \mathcal{N}_0(x_0\cdot\overline{x})(S)$$

is mapped to the object (31) under the morphism $\mathrm{st}^{\mathrm{I}-}_{x_0\cdot\overline{x}}$. Hence $\mathrm{st}^{\mathrm{I}+}_{x_0\cdot x}\left(\mathcal{N}_0(x_0\cdot x)\right) \subset \mathrm{st}^{\mathrm{I}-}_{x_0\cdot\overline{x}}\left(\mathcal{N}_0(x_0\cdot\overline{x})\right)$ as closed formal subschemes. The converse direction $\mathrm{st}^{\mathrm{I}+}_{x_0\cdot x}\left(\mathcal{N}_0(x_0\cdot x)\right) \supset \mathrm{st}^{\mathrm{I}-}_{x_0\cdot\overline{x}}\left(\mathcal{N}_0(x_0\cdot\overline{x})\right)$ can be proved similarly. Hence (a) is true. The idea of the proof of (b) is similar so we omit it. □

4.10. **Rapoport–Zink space with level $\Gamma_0(p)\times_{\mathbb{G}_m}\Gamma_0(p)$.** Let $x_0 \in \mathbb{B}$ be an element such that $\nu_p(q(x_0)) = 1$. Let $\pi : \mathcal{M} \to \mathcal{N}(x_0)$ be the blow-up morphism of the formal scheme $\mathcal{N}(x_0)$ along its unique closed point.

Let $\mathrm{Exc}_{\mathcal{M}}$ be the exceptional divisor on the formal scheme $\mathcal{M}$. Let $\mathcal{M}_{\mathbb{F}} := \mathcal{M}\times_W \mathbb{F}$ be the reduction mod p of the formal scheme $\mathcal{M}$.

Notice that the unique closed point pulls back to the unique closed point of the closed formal subschemes $\mathcal{N}(x_0)^{\mathrm{FF}}, \mathcal{N}(x_0)^{\mathrm{VV}}, \mathcal{N}(x_0)^{\mathrm{FV}}$ and $\mathcal{N}(x_0)^{\mathrm{VF}}$ of $\mathcal{N}(x_0)$. Let $\mathcal{M}^{\mathrm{FF}}, \mathcal{M}^{\mathrm{VV}}, \mathcal{M}^{\mathrm{FV}}$ and $\mathcal{M}^{\mathrm{VF}}$ be the strict transforms of the formal schemes $\mathcal{N}(x_0)^{\mathrm{FF}}, \mathcal{N}(x_0)^{\mathrm{VV}}, \mathcal{N}(x_0)^{\mathrm{FV}}$ and $\mathcal{N}(x_0)^{\mathrm{VF}}$ respectively under the blow up morphism $\pi : \mathcal{M} \to \mathcal{N}(x_0)$. The four closed formal subschemes $\mathcal{M}^{\mathrm{FF}}, \mathcal{M}^{\mathrm{VV}}, \mathcal{M}^{\mathrm{FV}}$ and $\mathcal{M}^{\mathrm{VF}}$ of $\mathcal{M}$ are all isomorphic to the formal scheme $\mathrm{Bl}_{(0,0)}\widehat{\mathbb{A}}^2_{\mathbb{F}}$.

Let $x \in \mathbb{B}$ be an element such that $\nu_p(q(x)) \geqslant 1$. The unique closed point of $\mathcal{N}(x_0)$ also pulls back to the unique closed point of the formal subschemes $\mathcal{N}_0(x)$ through the four closed immersion $\mathrm{st}^{\mathrm{I}+}_x, \mathrm{st}^{\mathrm{I}-}_x, \mathrm{st}^{\mathrm{II}+}_x, \mathrm{st}^{\mathrm{II}-}_x : \mathcal{N}_0(x) \to \mathcal{N}(x_0)$. Denote by $\mathcal{N}_0^{\mathrm{I}+}(x), \mathcal{N}_0^{\mathrm{I}-}(x), \mathcal{N}_0^{\mathrm{II}+}(x)$ and $\mathcal{N}_0^{\mathrm{II}-}(x)$ the strict transform of the formal scheme $\mathcal{N}_0(x)$ under the three morphisms $\mathrm{st}^{\mathrm{I}+}_x, \mathrm{st}^{\mathrm{I}-}_x, \mathrm{st}^{\mathrm{II}+}_x$ and $\mathrm{st}^{\mathrm{II}-}_x$ respectively. They are closed formal schemes of the formal scheme $\mathcal{M}$, and are all isomorphic to the formal scheme $\widetilde{\mathcal{N}}_0(x)$.

4.11. **An open cover of the Rapoport–Zink space $\mathcal{M}$.** We give a detailed description of the formal scheme $\mathcal{M}$ by giving an explicit open cover of it. By [KM85, Theorem 13.4.6], the local ring $\mathcal{O}_{\mathcal{N}(x_0)}$ is isomorphic to $W[[x_1, y_1, x_2, y_2]]/(p + x_1y_1, p + x_2y_2)$. The maximal ideal is generated by the images of x_1, y_1, x_2 and y_2. Then the blow up formal scheme $\mathcal{M}$ is covered by the following 4 open formal subschemes:

$\mathcal{M}_1^+$: Let $y_1 = u_{11}x_1, x_2 = v_{21}x_1$ and $y_2 = u_{21}x_1$. Then $u_{11} = u_{21}v_{21}$. Define

$$\mathcal{M}_1^+ = \mathrm{Spf}\, W[u_{21}, v_{21}][[x_1]]/(p + u_{21}v_{21}x_1^2).$$

$\mathcal{M}_2^+$: Let $x_1 = v_{12}x_2, y_1 = u_{12}x_2$ and $y_2 = u_{22}x_2$. Then $u_{22} = v_{12}u_{12}$. Define

$$\mathcal{M}_2^+ = \mathrm{Spf}\, W[v_{12}, u_{12}][[x_2]]/(p + v_{12}u_{12}x_2^2).$$

$\mathcal{M}_1^-$: Let $x_1 = w_{11}y_1, x_2 = w_{21}y_1$ and $y_2 = t_{21}y_1$. Then $w_{11} = w_{21}t_{21}$. Define

$$\mathcal{M}_1^- = \mathrm{Spf}\, W[w_{21}, t_{21}][[y_1]]/(p + w_{21}t_{21}y_1^2).$$

$\mathcal{M}_2^-$: Let $x_1 = w_{12}y_2, y_1 = t_{12}y_2$ and $x_2 = w_{22}y_2$. Then $w_{22} = w_{12}t_{12}$. Define

$$\mathcal{M}_2^- = \mathrm{Spf}\, W[w_{12}, t_{12}][[y_2]]/(p + w_{12}t_{12}y_2^2).$$

Notice that we have the following relations (on the intersections of the corresponding opens):

$$u_{21}=u_{12},\ v_{21}v_{12}=1,\ w_{21}u_{21}=1,\ t_{21}v_{21}=1,\ u_{21}w_{12}=1,\ t_{12}=v_{21}. \tag{32}$$

We denote this cover by $\mathcal{C}$. Let $\mathcal{M}^+=\mathcal{M}_1^+\cup\mathcal{M}_2^+$ and $\mathcal{M}^-=\mathcal{M}_1^-\cup\mathcal{M}_2^-$.

Proposition 4.11.1. *The following facts hold:*

(i) *The 3-dimensional formal scheme $\mathcal{M}$ is regular.*
(ii) *The exceptional divisor $\mathrm{Exc}_{\mathcal{M}}$ is isomorphic to $\mathbb{P}^1_{\mathbb{F}}\times\mathbb{P}^1_{\mathbb{F}}$.*
(iii) *The closed formal subschemes $\mathcal{M}^{\mathrm{FF}},\mathcal{M}^{\mathrm{VV}},\mathcal{M}^{\mathrm{FV}}$ and $\mathcal{M}^{\mathrm{VF}}$ are regular Cartier divisors on $\mathcal{M}$, and we have the following equality of Cartier divisors:*

$$\mathcal{M}_{\mathbb{F}}=2\cdot\mathrm{Exc}_{\mathcal{M}}+\mathcal{M}^{\mathrm{FF}}+\mathcal{M}^{\mathrm{VV}}+\mathcal{M}^{\mathrm{FV}}+\mathcal{M}^{\mathrm{VF}}.$$

(iv) *Let $x\in\mathbb{B}$ be an element such that $\nu_p(q(x))\geqslant 1$, the closed formal subschemes $\mathcal{N}_0^{\mathrm{I}+}(x),\mathcal{N}_0^{\mathrm{I}-}(x)$, $\mathcal{N}_0^{\mathrm{II}+}(x)$ and $\mathcal{N}_0^{\mathrm{II}-}(x)$ are regular Cartier divisors on $\mathcal{M}$.*

Proof. The four open formal subschemes $\mathcal{M}_1^+,\mathcal{M}_2^+,\mathcal{M}_1^-$ and $\mathcal{M}_2^-$ of the formal scheme $\mathcal{M}$ cover $\mathcal{M}$ and are all regular formal schemes of dimension 3, hence (i) is true. Notice that the formal scheme $\mathrm{Bl}_{(0,0)}\widehat{\mathbb{A}}^2_{\mathbb{F}}$ and $\widetilde{\mathcal{N}}_0(x)$ are 2-dimensional regular formal schemes, hence the regular closed formal subschemes $\mathcal{M}^{\mathrm{FF}},\mathcal{M}^{\mathrm{VV}},\mathcal{M}^{\mathrm{FV}},\mathcal{M}^{\mathrm{VF}},\mathcal{N}_0^{\mathrm{I}+}(x),\mathcal{N}_0^{\mathrm{I}-}(x),\mathcal{N}_0^{\mathrm{II}+}(x),\mathcal{N}_0^{\mathrm{II}-}(x)$ of $\mathcal{M}$ must be regular Cartier divisors on $\mathcal{M}$, i.e., (iv) and the first part of (iii) are true.

The exceptional divisor $\mathrm{Exc}_{\mathcal{M}}$ is covered by the following 4 open subschemes

$$\mathrm{Exc}^+_{\mathcal{M},1}:=\mathrm{Exc}_{\mathcal{M}}\cap\mathcal{M}_1^+\simeq\mathrm{Spec}\,\mathbb{F}[u_{21},v_{21}],\ \mathrm{Exc}^+_{\mathcal{M},2}:=\mathrm{Exc}_{\mathcal{M}}\cap\mathcal{M}_2^+\simeq\mathrm{Spec}\,\mathbb{F}[v_{12},u_{12}],$$
$$\mathrm{Exc}^-_{\mathcal{M},1}:=\mathrm{Exc}_{\mathcal{M}}\cap\mathcal{M}_1^+\simeq\mathrm{Spec}\,\mathbb{F}[w_{21},t_{21}],\ \mathrm{Exc}^-_{\mathcal{M},2}:=\mathrm{Exc}_{\mathcal{M}}\cap\mathcal{M}_2^-\simeq\mathrm{Spec}\,\mathbb{F}[w_{12},t_{12}].$$

By the relations between the coordinates in (32), the above 4 schemes glue together and the resulting scheme $\mathbb{P}^1_{\mathbb{F}}\times\mathbb{P}^1_{\mathbb{F}}$. Hence (ii) is true.

Now we prove the second part of (iii). By §4.6, we can assume that the irreducible component $\mathcal{N}_0(x_0)^{\mathrm{F}}$ is given by the equation $x=0$, while the irreducible component $\mathcal{N}_0(x_0)^{\mathrm{F}}$ is given by the equation $y=0$. Hence the equations of the four closed formal subschemes $\mathcal{N}(x_0)^{\mathrm{FF}},\mathcal{N}(x_0)^{\mathrm{VV}},\mathcal{N}(x_0)^{\mathrm{FV}}$ and $\mathcal{N}(x_0)^{\mathrm{VF}}$ are given by

$$\mathcal{N}(x_0)^{\mathrm{FF}}:(y_1,y_2),\ \ \mathcal{N}(x_0)^{\mathrm{VV}}:(x_1,x_2),\ \ \mathcal{N}(x_0)^{\mathrm{FV}}:(y_1,x_2),\ \ \mathcal{N}(x_0)^{\mathrm{VF}}:(x_1,y_2).$$

Hence it's easy to see that

$\mathcal{M}^{\mathrm{FF}}$: $\mathcal{M}^{\mathrm{FF}}\subset\mathcal{M}_1^+\cup\mathcal{M}_2^+$ is cut out by the equation $u_{21}=0$ on $\mathcal{M}_1^+$ and $u_{12}=0$ on $\mathcal{M}_2^+$;
$\mathcal{M}^{\mathrm{VV}}$: $\mathcal{M}^{\mathrm{VV}}\subset\mathcal{M}_1^-\cup\mathcal{M}_2^-$ is cut out by the equation $w_{21}=0$ on $\mathcal{M}_1^-$ and $w_{12}=0$ on $\mathcal{M}_2^-$;
$\mathcal{M}^{\mathrm{FV}}$: $\mathcal{M}^{\mathrm{FV}}\subset\mathcal{M}_1^+\cup\mathcal{M}_2^-$ is cut out by the equation $v_{21}=0$ on $\mathcal{M}_1^+$ and $t_{12}=0$ on $\mathcal{M}_2^-$;
$\mathcal{M}^{\mathrm{VF}}$: $\mathcal{M}^{\mathrm{VF}}\subset\mathcal{M}_2^+\cup\mathcal{M}_1^-$ is cut out by the equation $v_{12}=0$ on $\mathcal{M}_2^+$ and $t_{21}=0$ on $\mathcal{M}_1^-$.

Therefore the second part of (iii) is true by the explicit open covering of the formal scheme $\mathcal{M}$ by $\mathcal{M}_1^+,\mathcal{M}_2^+,\mathcal{M}_1^-$ and $\mathcal{M}_2^-$. □

Remark 4.11.2. For simplicity, in the following paragraphs we denote u_{21} by u, denote v_{21} by v. Then the open covering of the exceptional divisor $\mathrm{Exc}_{\mathcal{M}}$ can be written in the following way by the relations in (32)

$$\mathrm{Exc}^+_{\mathcal{M},1}\simeq\mathrm{Spec}\,\mathbb{F}[v,u],\ \mathrm{Exc}^+_{\mathcal{M},2}\simeq\mathrm{Spec}\,\mathbb{F}[v',u],$$

$$\mathrm{Exc}^{-}_{\mathcal{M},1} \simeq \operatorname{Spec} \mathbb{F}[v', u'], \ \mathrm{Exc}^{-}_{\mathcal{M},2} \simeq \operatorname{Spec} \mathbb{F}[v, u'].$$

Notice that the coordinate v' (resp. u') satisfies $v \cdot v' = 1$ (resp. $u \cdot u' = 1$) when v (resp. u) is nonzero.

Use the coordinates u, v, u' and v'. We fix an isomorphism $\iota : \mathrm{Exc}_{\mathcal{M}} \simeq \mathbb{P}^1_{\mathbb{F}} \times \mathbb{P}^1_{\mathbb{F}}$ such that the first $\mathbb{P}^1_{\mathbb{F}}$ is obtained by gluing $\operatorname{Spec} \mathbb{F}[v]$ and $\operatorname{Spec} \mathbb{F}[v']$ along $\operatorname{Spec} \mathbb{F}[v, v'] \simeq \operatorname{Spec} \mathbb{F}[v, v^{-1}]$, while the second $\mathbb{P}^1_{\mathbb{F}}$ is obtained by gluing $\operatorname{Spec} \mathbb{F}[u]$ and $\operatorname{Spec} \mathbb{F}[u']$ along $\operatorname{Spec} \mathbb{F}[u, u'] \simeq \operatorname{Spec} \mathbb{F}[u, u^{-1}]$.

4.12. **An automorphism of $\mathcal{M}$.** Let $\iota_0 : \mathcal{N}_0(x_0) \to \mathcal{N}_0(x_0)$ be the following morphism: Let $S \in \mathrm{Nilp}_W$. Let $c_0 = \overline{x}_0 \cdot x_0^{-1} \in \mathbb{B}$. For an object $\left(X \xrightarrow{\pi} X', (\rho, \rho')\right) \in \mathcal{N}_0(x_0)(S)$,

$$\iota_0 : \left(X \xrightarrow{\pi} X', (\rho, \rho')\right) \mapsto \left(X' \xrightarrow{\pi^\vee} X, (\rho', \rho \circ c_0)\right).$$

Here we are identifying X^{univ} (resp. X'^{univ}) with $(X^{\mathrm{univ}})^\vee$ (resp. $(X'^{\mathrm{univ}})^\vee$) via the principal polarization of X^{univ} (resp. X'^{univ}). The morphism ι_0 is an automorphism of $\mathcal{N}_0(x_0)$ over W (but not an involution unless $\overline{x}_0 = x_0$). It induces an automorphism $\mathrm{id} \times \iota_0 : \mathcal{N}(x_0) \to \mathcal{N}(x_0)$ of the formal scheme $\mathcal{N}(x_0)$. Under this isomorphism, the universal pair (30) is mapped to

$$\left(X_1^{\mathrm{univ}} \xrightarrow{x_{0,1}^{\mathrm{univ}}} X_1'^{\mathrm{univ}}, \left(\rho_1^{\mathrm{univ}}, \rho_1'^{\mathrm{univ}}\right)\right), \quad \left(X_2'^{\mathrm{univ}} \xrightarrow{(x_{0,2}^{\mathrm{univ}})^\vee} X_2^{\mathrm{univ}}, \left(\rho_2'^{\mathrm{univ}}, \rho_2^{\mathrm{univ}} \circ c_0\right)\right).$$

By the moduli interpretations of special cycles on $\mathcal{N}(x_0)$, we have that for all subsets $H \subset \mathbb{B}$,

$$\mathcal{Y}_{\mathcal{N}(x_0)}(H) = (\mathrm{id} \times \iota_0)^* \left(\mathcal{Z}_{\mathcal{N}(x_0)}(x_0 \cdot H)\right) := \mathcal{Z}_{\mathcal{N}(x_0)}(x_0 \cdot H) \times_{\mathcal{N}(x_0), \mathrm{id} \times \iota_0} \mathcal{N}(x_0). \tag{33}$$

Let $x \in \mathbb{B}$ be an element such that $\nu_p(q(x)) \geqslant 1$. By the definition of the morphisms $\mathrm{st}_x^{\mathrm{I}+}$,$\mathrm{st}_x^{\mathrm{I}-}$, $\mathrm{st}_x^{\mathrm{II}+}$ and $\mathrm{st}_x^{\mathrm{II}-} : \mathcal{N}_0(x) \to \mathcal{N}(x_0)$ in §4.8, we have the following identities of closed formal subschemes of $\mathcal{N}(x_0)$:

$$\mathrm{st}_{c_0 x}^{\mathrm{II}+}\left(\mathcal{N}_0(c_0 x)\right) = (\mathrm{id} \times \iota_0)^* \left(\mathrm{st}_x^{\mathrm{I}+}\left(\mathcal{N}_0(x)\right)\right), \ \mathrm{st}_{\overline{x}}^{\mathrm{II}-}\left(\mathcal{N}_0(\overline{x})\right) = (\mathrm{id} \times \iota_0)^* \left(\mathrm{st}_x^{\mathrm{I}-}\left(\mathcal{N}_0(x)\right)\right), \tag{34}$$

$$\mathrm{st}_x^{\mathrm{I}+}\left(\mathcal{N}_0(x)\right) = (\mathrm{id} \times \iota_0)^* \left(\mathrm{st}_x^{\mathrm{II}+}\left(\mathcal{N}_0(x)\right)\right), \ \mathrm{st}_{\overline{c_0 x}}^{\mathrm{I}-}\left(\mathcal{N}_0(\overline{c_0 x})\right) = (\mathrm{id} \times \iota_0)^* \left(\mathrm{st}_x^{\mathrm{II}-}\left(\mathcal{N}_0(x)\right)\right). \tag{35}$$

Notice that by the universal property of the blow-up morphism $\pi : \mathcal{M} \to \mathcal{N}(x_0)$, the automorphism $\mathrm{id} \times \iota_0$ induces an automorphism $\iota^{\mathcal{M}}$ of the formal scheme $\mathcal{M}$. By (34) and (35), we have the following identities of closed formal subschemes of $\mathcal{M}$:

$$\mathcal{N}_0^{\mathrm{II}+}(c_0 x) = \left(\iota^{\mathcal{M}}\right)^* \left(\mathcal{N}_0^{\mathrm{I}+}(x)\right), \ \mathcal{N}_0^{\mathrm{II}-}(\overline{x}) = \left(\iota^{\mathcal{M}}\right)^* \left(\mathcal{N}_0^{\mathrm{I}-}(x)\right), \tag{36}$$

$$\mathcal{N}_0^{\mathrm{I}+}(x) = \left(\iota^{\mathcal{M}}\right)^* \left(\mathcal{N}_0^{\mathrm{II}+}(x)\right), \ \mathcal{N}_0^{\mathrm{I}-}(\overline{c_0 x}) = \left(\iota^{\mathcal{M}}\right)^* \left(\mathcal{N}_0^{\mathrm{II}-}(x)\right). \tag{37}$$

4.13. **Special cycles on the Rapoport–Zink space $\mathcal{M}$.**

Definition 4.13.1. For a subset $H \subset \mathbb{B}$, define the special cycles $\mathcal{Z}_{\mathcal{M}}(H) := \mathcal{Z}_{\mathcal{N}(x_0)}(H) \times_{\mathcal{N}(x_0)} \mathcal{M}$ and $\mathcal{Y}_{\mathcal{M}}(H) := \mathcal{Y}_{\mathcal{N}(x_0)}(H) \times_{\mathcal{N}(x_0)} \mathcal{M}$, i.e., we have the following Cartesian diagrams

$$\begin{array}{ccc} \mathcal{Z}_{\mathcal{M}}(H) & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \pi \\ \mathcal{Z}_{\mathcal{N}(x_0)}(H) & \longrightarrow & \mathcal{N}(x_0), \end{array} \qquad \begin{array}{ccc} \mathcal{Y}_{\mathcal{M}}(H) & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \pi \\ \mathcal{Y}_{\mathcal{N}(x_0)}(H) & \longrightarrow & \mathcal{N}(x_0), \end{array}$$

Since $\mathcal{Y}_{\mathcal{N}(x_0)}(H) = (\mathrm{id} \times \iota_0)^* \left(\mathcal{Z}_{\mathcal{N}(x_0)}(x_0 \cdot H)\right)$ and $\mathcal{Z}_{\mathcal{N}(x_0)}(\overline{x_0} \cdot H) = (\mathrm{id} \times \iota_0)^* \left(\mathcal{Y}_{\mathcal{N}(x_0)}(H)\right)$, we have $\mathcal{Y}_{\mathcal{M}}(H) = \left(\iota^{\mathcal{M}}\right)^* \left(\mathcal{Z}_{\mathcal{M}}(x_0 \cdot H)\right)$ and $\mathcal{Z}_{\mathcal{M}}(\overline{x_0} \cdot H) = \left(\iota^{\mathcal{M}}\right)^* \left(\mathcal{Y}_{\mathcal{M}}(H)\right)$.

By the moduli interpretation of the formal scheme $\mathcal{N}(x_0)$, the morphism π gives rise to the following two pairs of deformations of the quasi-isogeny x_0 over the formal scheme $\mathcal{M}$:

$$\left(X_{1,\mathcal{M}} \stackrel{(x_0)_{1,\mathcal{M}}}{\longrightarrow} X'_{1,\mathcal{M}}, (\rho_{1,\mathcal{M}}, \rho'_{1,\mathcal{M}})\right), \quad \left(X_{2,\mathcal{M}} \stackrel{(x_0)_{2,\mathcal{M}}}{\longrightarrow} X'_{2,\mathcal{M}}, (\rho_{2,\mathcal{M}}, \rho'_{2,\mathcal{M}})\right). \tag{38}$$

By the moduli interpretation of the special cycle $\mathcal{Z}_{\mathcal{N}(x_0)}(H)$ in Definition 4.9.1, the special cycle $\mathcal{Z}_{\mathcal{M}}(H)$ is cut out by the following conditions,

$$\rho_{2,\mathcal{M}} \circ x \circ (\rho_{1,\mathcal{M}})^{-1} \in \mathrm{Hom}(X_{1,\mathcal{M}}, X_{2,\mathcal{M}});$$
$$\rho'_{2,\mathcal{M}} \circ x' \circ (\rho'_{1,\mathcal{M}})^{-1} \in \mathrm{Hom}(X'_{1,\mathcal{M}}, X'_{2,\mathcal{M}}).$$

for all $x \in H$.

For simplicity, we denote the special cycle $\mathcal{Z}_{\mathcal{M}}(H)$ by $\mathcal{Z}(H)$ in all the following paragraphs. If $H = \{x\}$ where $x \in \mathbb{B}$, denote the special cycle $\mathcal{Z}(\{x\})$ by $\mathcal{Z}(x)$.

Recall that we define $\mathcal{M}^+ = \mathcal{M}_1^+ \cup \mathcal{M}_2^+$ and $\mathcal{M}^- = \mathcal{M}_1^- \cup \mathcal{M}_2^-$. For a symbol $? \in \{+, -\}$, let the pair

$$\left((X_{1,\mathcal{M}^?} \stackrel{(x_0)_{1,\mathcal{M}^?}}{\longrightarrow} X'_{1,\mathcal{M}^?}), (\rho_{1,\mathcal{M}^?}, \rho'_{1,\mathcal{M}^?})\right), \quad \left((X_{2,\mathcal{M}^?} \stackrel{(x_0)_{2,\mathcal{M}^?}}{\longrightarrow} X'_{2,\mathcal{M}^?}), (\rho_{2,\mathcal{M}^?}, \rho'_{2,\mathcal{M}^?})\right)$$

be the base change of the universal pair (38) over $\mathcal{M}$ to $\mathcal{M}^?$. Define $\mathcal{Z}^?(x) := \mathcal{Z}(x) \cap \mathcal{M}^?$. Let $p_? : \mathcal{M}^? \to \mathcal{N}$ be the following composition (recall that s_+, s_- are defined in (27) and (28)):

$$p_? : \mathcal{M}^? \to \mathcal{M} \xrightarrow{\pi} \mathcal{N}(x_0) \xrightarrow{s_?} \mathcal{N}. \tag{39}$$

Lemma 4.13.2. *Let $x \in \mathbb{B}$ be a nonzero element (recall that $x' = x_0 \cdot x \cdot x_0^{-1}$). The closed formal subscheme $\mathcal{Z}(x)$ of the formal scheme $\mathcal{M}$ is cut out by the conditions*

- (+) *On $\mathcal{M}^+$: $\rho_{2,\mathcal{M}^+} \circ x \circ (\rho_{1,\mathcal{M}^+})^{-1} \in \mathrm{Hom}(X_{1,\mathcal{M}^+}, X_{2,\mathcal{M}^+})$, i.e.,*
$$\mathcal{Z}^+(x) = \mathcal{Z}_{\mathcal{N}}(x) \times_{\mathcal{N}, p_+} \mathcal{M}^+.$$
- (−) *On $\mathcal{M}^-$: $\rho_{2,\mathcal{M}^-} \circ x' \circ (\rho_{1,\mathcal{M}^-})^{-1} \in \mathrm{Hom}(X_{1,\mathcal{M}^-}, X_{2,\mathcal{M}^-})$, i.e.,*
$$\mathcal{Z}^-(x) = \mathcal{Z}_{\mathcal{N}}(x') \times_{\mathcal{N}, p_-} \mathcal{M}^-.$$

Therefore the special cycle $\mathcal{Z}(x)$ is a Cartier divisor on $\mathcal{M}$.

Proof. We first prove the (+) case. Let $z \in \mathcal{Z}^+(x)(\mathbb{F})$ be a point. Let R be the completed local ring of the formal scheme $\mathcal{M}$ at z. Without loss of generality, we can assume that $z \in \mathcal{M}_1^+(\mathbb{F})$. Let $I \subset R$ be the ideal cutting out the divisor $\mathcal{Z}(x)$, we will show that I is a principal ideal.

By the moduli interpretation of $\mathcal{Z}(x)$, the ideal I is generated by two elements f_x and $f_{x'}$, where f_x (resp. $f_{x'}$) is the equation of the universal closed formal subscheme of $\mathcal{M}$ over which the quasi-isogeny x (resp. x') lifts to an isogeny. By Lemma 4.3.2, the elements f_x and $f_{x'}$ are nonzero when $x \neq 0$, hence $I \neq 0$. The surjective homomorphism $R/I^2 \to R/I$ is equipped with a nilpotent pd structure. We have the following commutative diagram,

$$\begin{array}{ccc} \mathbb{D}(X_1) & \xrightarrow{\mathbb{D}(x)} & \mathbb{D}(X_2) \\ {\scriptstyle \mathbb{D}(x_0)}\downarrow & & \downarrow{\scriptstyle \mathbb{D}(x_0)} \\ \mathbb{D}(X_1') & \xrightarrow{\mathbb{D}(x')} & \mathbb{D}(X_2'). \end{array}$$

Let's use the specific basis of the four Dieudonne modules stated in Proposition 4.6.2. Then

$$\mathbb{D}(x)[e_1, f_1] = [e_2, f_2] \begin{pmatrix} a & -pc' \\ c & d \end{pmatrix}, \quad \mathbb{D}(x')[e_1', f_1'] = [e_2', f_2'] \begin{pmatrix} a' & -pc \\ c' & a \end{pmatrix},$$

where $a, a', c, c', d \in R/I^2$ and $p(a' - d) = 0 \in R/I^2$. Then

$$\mathbb{D}(x)(f_1 + x_1 e_1) = (d + cx_1) f_2 + (ax_1 - pc')e_2,$$
$$\mathbb{D}(x')(f_1' + y_1 e_1') = (a + c'y_1) f_2' + (a'y_1 - pc)e_2'.$$

Since $z \in \mathcal{M}_1^+$, we use the coordinate for $\mathcal{M}_1^+$ in §4.11. Recall that we have $y_1 = ux_1$ and $x_2 = vx_1$. The filtration on the Dieudonne crystals are given explicitly in Proposition 4.6.2, hence

$$f_x = cvx_1^2 + dvx_1 - ax_1 - c'uvx_1^2 \in R/I^2;$$
$$f_{x'} = c'u^2vx_1^2 + aux_1 - a'uvx_1 - cuvx_1^2 = -uf_x + uvx_1(d - a') \in R/I^2.$$

Notice that $x_1 | f_x, f_{x'}$, let $g = f_x/x_1$ and $g' = f_{x'}/x_1$ (In Proposition 5.3.1, we calculated the equations of f_x and $f_{x'}$ which imply that both g and g' are not units in R). Since $p = -uvx_1^2$ and $p(a' - d) = 0 \in R/I^2$, we have

$$uvx_1(d - a') \in (gf_x, g'f_{x'}, gf_{x'}).$$

Then $f_{x'} - uf_x \in (gf_x, g'f_{x'}, gf_{x'}) + I^2$. Therefore $f_{x'} = t \cdot f_x$ for some element $t \in R$. Hence $I = (f_x, f_{x'}) = (f_x)$ is a principal ideal, and $\mathcal{Z}^+(x) = \mathcal{Z}_{\mathcal{N}}(x) \times_{\mathcal{N}, p_+} \mathcal{M}^+$. The proof of the $(-)$ case is similar so we omit the details here. □

By the identity $\mathcal{Y}(x) = \left(\iota^{\mathcal{M}}\right)^* (\mathcal{Z}(x_0 \cdot x))$, we obtain the following

Corollary 4.13.3. *Let $x \in \mathbb{B}$ be a nonzero element. The closed formal subscheme $\mathcal{Y}(x)$ is a Cartier divisor on $\mathcal{M}$.*

In the following paragraphs, for a nonzero element $x \in \mathbb{B}$, we refer to $\mathcal{Z}(x)$ as a special divisor on the formal scheme $\mathcal{M}$. By the moduli interpretation of the special divisor $\mathcal{Z}(x)$ in Proposition 4.13.2, there is a closed immersion $\mathcal{Z}(p^{-1}x) \to \mathcal{Z}(x)$.

Definition 4.13.4. Let $x \in \mathbb{B}$ be a nonzero element. Define the difference divisor associated to x to be the following effective Cartier divisor on the formal scheme $\mathcal{M}$,

$$\mathcal{D}(x) = \mathcal{Z}(x) - \mathcal{Z}(p^{-1}x).$$

For a symbol $? \in \{+, -\}$, define $\mathcal{D}^?(x) = \mathcal{D}(x) \cap \mathcal{M}^?$.

Remark 4.13.5. By Proposition 4.13.2, we have

$$\mathcal{D}^+(x) = \mathcal{N}_0(x) \times_{\mathcal{N}, p_+} \mathcal{M}^+ \text{ and } \mathcal{D}^+(x) = \mathcal{N}_0(x') \times_{\mathcal{N}, p_-} \mathcal{M}^-.$$

By the isomorphism $\mathcal{D}_{\mathcal{N}}(x) \simeq \mathcal{N}_0(x)$ in Lemma 4.4.1, we can give a moduli interpretation for the difference divisor $\mathcal{D}(x)$ as follows:

$$\mathcal{D}^+(x) : \text{the quasi-isogeny} \rho_{2,\mathcal{M}^+} \circ x \circ (\rho_{1,\mathcal{M}^+})^{-1} \text{lifts to a cyclic isogeny},$$
$$\mathcal{D}^-(x) : \text{the quasi-isogeny} \rho_{2,\mathcal{M}^-} \circ x' \circ (\rho_{1,\mathcal{M}^-})^{-1} \text{lifts to a cyclic isogeny}.$$

5. Special divisor and the exceptional divisor

5.1. **Line bundles on the exceptional divisor $\mathrm{Exc}_{\mathcal{M}}$.** Let $(X, \mathcal{O}_X)$ be a noetherian formal scheme over $\mathrm{Spf}\, W$, we say a noetherian formal scheme $(X, \mathcal{O}_X)$ over $\mathrm{Spf}\, W$ is of pure dimension d if for every closed point $x \in X(\mathbb{F})$, we have

$$\dim \mathcal{O}_{X,x} = d.$$

Now we assume $(X, \mathcal{O}_X)$ is a regular noetherian formal scheme over $\mathrm{Spf}\, W$ of pure dimension d, then we have a natural isomorphism [Zha21, §B.1] (see also [GS87, Lemma 1.9], the proof also works here)

$$K_0^Y(X) \xrightarrow{\sim} K'(Y). \tag{40}$$

Let $Y \subset X$ is a closed formal subscheme. If $Z \subset Y$ is a closed subscheme such that $\mathrm{codim}_X Z \geqslant d-i$, therefore $\dim Y \leqslant i$ by the definition of dimension and codimension in §1.7.2 and the assumption that X is of pure dimension d. Therefore the isomorphism (40) identifies

$$\mathrm{F}^{d-i} K_0^Y(X) \xrightarrow{\sim} \mathrm{F}_i K'(Y) \tag{41}$$

Lemma 5.1.1. *Let $(X, \mathcal{O}_X)$ be a regular Noetherian formal scheme over $\mathrm{Spf}\, W$ of pure dimension d.*

(a). *Let $C, D \subset X$ be two effective Cartier divisors on X intersecting properly, we have the following equality in $\mathrm{Gr}^1 K_0^{C\cup D}(X)$,*

$$[\mathcal{O}_{C+D}] = [\mathcal{O}_C] + [\mathcal{O}_D].$$

(b). *Let $Z \subset X$ be a divisor such that Z itself is an integral regular Noetherian scheme, we have the following equality in $\mathrm{Gr}^1 K_0^Z(X)$,*

$$[\mathcal{O}_{nZ}] = n \cdot [\mathcal{O}_Z] \text{ for all } n > 0.$$

Proof. Part (a) follows from [Zha21, Lemma B.1]. In our case, we can take $X_1 = C$, $X_2 = D$ and $[\mathcal{E}] = [\mathcal{O}_{C+D}]$, then *loc. cit.* asserts that we have the identity

$$[\mathcal{O}_{C+D}] = [\mathcal{O}_C] + [\mathcal{O}_D]$$

in the group $K_0'(C \cup D)/K_0'(C \cap D) \simeq K_0^{C\cup D}(X)/K_0^{C\cap D}(X)$. Since the two divisors C and D intersect properly, we have $K_0^{C\cap D}(X) \subset \mathrm{F}^2 K_0^{C\cap D}(X)$, therefore we have the claimed identity in (a).

Now we prove (b). Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of the closed scheme Z. By [Sta25, 0FD0], we have

$$[\mathcal{O}_Z] = [\mathcal{I}^k/\mathcal{I}^{k+1}] \bmod \mathrm{F}_{d-2} K_0'(Z), \;\; k \in \mathbb{Z}$$

because $\mathcal{I}^k/\mathcal{I}^{k+1}$ is a line bundle on the integral regular scheme Z whose dimension is $d-1$. Since X is regular of pure dimension d, the subgroup $\mathrm{F}^2 K_0^Z(X)$ is isomorphic to $\mathrm{F}_{d-2} K_0'(Z)$ by (41). Then we have the following identity in $K_0^Z(X)$:

$$[\mathcal{O}_{nZ}] = [\mathcal{O}_Z] + [\mathcal{I}/\mathcal{I}^2] + \cdots + [\mathcal{I}^{n-1}/\mathcal{I}^n] = [\mathcal{O}_Z] + [\mathcal{O}_Z] + \cdots + [\mathcal{O}_Z] = n[\mathcal{O}_Z] \bmod \mathrm{F}^2K_0^Z(X).$$

Therefore we have the identity in (b). □

Under the isomorphism $\iota : \mathrm{Exc}_{\mathcal{M}} \simeq \mathbb{P}^1_{\mathbb{F}} \times \mathbb{P}^1_{\mathbb{F}}$ we fixed in Remark 4.11.2, let $\mathrm{pr}_1 : \mathrm{Exc}_{\mathcal{M}} \to \mathbb{P}^1_{\mathbb{F}}$ be the first projection, while $\mathrm{pr}_2 : \mathrm{Exc}_{\mathcal{M}} \to \mathbb{P}^1_{\mathbb{F}}$ be the second projection. Let $\mathcal{L}$ be a line bundle on $\mathrm{Exc}_{\mathcal{M}}$, it is isomorphic to $\mathrm{pr}_1^*\mathcal{O}(m) \otimes \mathrm{pr}_2^*\mathcal{O}(n)$ for some integers m and n. Define

$$\mathcal{O}(m,n) := \mathrm{pr}_1^*\mathcal{O}(m) \otimes \mathrm{pr}_2^*\mathcal{O}(n) \in \mathrm{Pic}(\mathrm{Exc}_{\mathcal{M}}).$$

For two line bundles $\mathcal{L}, \mathcal{G} \in \mathrm{Pic}(\mathrm{Exc}_{\mathcal{M}})$, denote by $\mathcal{L} \cdot \mathcal{G}$ the intersection product $\mathrm{Pic}(\mathrm{Exc}_{\mathcal{M}}) \times \mathrm{Pic}(\mathrm{Exc}_{\mathcal{M}}) \to \mathbb{Z}$ on the group $\mathrm{Pic}(\mathrm{Exc}_{\mathcal{M}})$.

Lemma 5.1.2. *Let m_1, m_2, n_1, n_2 be integers, then the intersection number of the line bundles $\mathcal{O}(m_1, n_1)$ and $\mathcal{O}(m_2, n_2)$ is*

$$\mathcal{O}(m_1, n_1) \cdot \mathcal{O}(m_2, n_2) = m_1n_2 + m_2n_1.$$

Proof. This follows from the fact that $\mathcal{O}(1,0) \cdot \mathcal{O}(0,1) = 1$ and the additivity of intersection numbers between line bundles. □

5.2. **Self-intersection of the exceptional divisor.** For $i = 0, 1, 2$, we have $\mathrm{F}^{3-i}K_0^{\mathrm{Exc}_{\mathcal{M}}}(\mathcal{M}) \simeq \mathrm{F}_iK_0(\mathrm{Exc}_{\mathcal{M}})$. Notice that

$$\begin{aligned}\mathrm{Gr}^2K_0^{\mathrm{Exc}_{\mathcal{M}}}(\mathcal{M}) &:= \mathrm{F}^2K_0^{\mathrm{Exc}_{\mathcal{M}}}(\mathcal{M})/\mathrm{F}^3K_0^{\mathrm{Exc}_{\mathcal{M}}}(\mathcal{M}) \\ &\simeq \mathrm{F}_1K_0(\mathrm{Exc}_{\mathcal{M}})/\mathrm{F}_0K_0(\mathrm{Exc}_{\mathcal{M}}) \simeq \mathrm{CH}^1(\mathrm{Exc}_{\mathcal{M}}) \simeq \mathrm{Pic}(\mathrm{Exc}_{\mathcal{M}}).\end{aligned}$$

The isomorphism is given by $\mathcal{L} \in \mathrm{Pic}(\mathrm{Exc}_{\mathcal{M}}) \longmapsto [\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}] - [\mathcal{L}] \in \mathrm{Gr}^2K_0^{\mathrm{Exc}_{\mathcal{M}}}(\mathcal{M})$. We remind the readers that we will use the notations in Remark 4.11.2 in the following paragraphs.

Lemma 5.2.1. *We have the following equality in the group $\mathrm{Gr}^2K_0^{\mathrm{Exc}_{\mathcal{M}}}(\mathcal{M}) \simeq \mathrm{Pic}(\mathrm{Exc}_{\mathcal{M}})$,*

$$[\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}] = \mathcal{O}(-1,-1).$$

Proof. Let $\mathcal{I} \subset \mathcal{O}_{\mathcal{M}}$ be the ideal sheaf of the exceptional ideal $\mathrm{Exc}_{\mathcal{M}}$. We know that $[\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}] = [\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}] - [\mathcal{I}/\mathcal{I}^2] \in \mathrm{F}_1K_0'(\mathrm{Exc}_{\mathcal{M}}) \simeq \mathrm{F}^2K_0^{\mathrm{Exc}_{\mathcal{M}}}(\mathcal{M})$ (see the proof of the Lemma 5.1.1 (b)). Under the isomorphism $\mathrm{Gr}^2K_0^{\mathrm{Exc}_{\mathcal{M}}}(\mathcal{M}) \simeq \mathrm{Pic}(\mathrm{Exc}_{\mathcal{M}})$, the element $[\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}]$ is given by restricting the invertible sheaf $\mathcal{I}$ to $\mathrm{Exc}_{\mathcal{M}}$. Under the open cover of $\mathcal{M}$ in §4.11, the divisor $\mathrm{Exc}_{\mathcal{M}}$ is given by the following equations and transformation rules:

$$\begin{array}{ccccccc} \mathcal{M}_1^+ & & \mathcal{M}_1^- & & \mathcal{M}_2^+ & & \mathcal{M}_2^- \\ \Cup & & \Cup & & \Cup & & \Cup \\ x_1 & \underset{\times(uv)^{-1}}{\overset{\times uv}{\rightleftarrows}} & y_1 & \underset{\times u}{\overset{\times u^{-1}}{\rightleftarrows}} & x_2 & \underset{\times vu^{-1}}{\overset{\times uv^{-1}}{\rightleftarrows}} & y_2 \end{array}$$

The same transformation rule also applies to the corresponding open cover of $\mathrm{Exc}_{\mathcal{M}} \simeq \mathbb{P}^1_{\mathbb{F}} \times \mathbb{P}^1_{\mathbb{F}}$ in Remark 4.11.2. Therefore the corresponding line bundle is $\mathcal{O}(-1,-1)$. □

5.3. **Intersections of special divisors and the exceptional divisor.** Let $x \in \mathbb{B}$ be a nonzero element such that $q(x) \in \mathbb{Z}_p$.

Proposition 5.3.1. *Let $x \in \mathbb{B}$ be a nonzero element such that $q(x) \in \mathbb{Z}_p$ such that $\nu_p(q(x)) = n$ for some integer $n \geqslant 0$. Then there exists a Cartier divisor $\widetilde{\mathcal{D}}(x)$ on the formal scheme $\mathcal{M}$ which intersects with the exceptional divisor $\mathrm{Exc}_{\mathcal{M}}$ properly and*

(i) *When $n = 0$, the following equality of Cartier divisors on $\mathcal{M}$ holds,*

$$\mathcal{D}(x) = \mathrm{Exc}_{\mathcal{M}} + \widetilde{\mathcal{D}}(x).$$

Moreover, the Cartier divisor $\mathrm{Exc}_{\mathcal{M}} \cap \widetilde{\mathcal{D}}(x)$ of $\mathrm{Exc}_{\mathcal{M}}$ is

$$\mathrm{Exc}_{\mathcal{M}} \cap \widetilde{\mathcal{D}}(x) = (v = \text{a nonzero number in } \mathbb{F})\,.$$

(ii) *When $n \geqslant 1$, the following equality of Cartier divisors on $\mathcal{M}$ holds,*

$$\mathcal{D}(x) = 2 \cdot \mathrm{Exc}_{\mathcal{M}} + \widetilde{\mathcal{D}}(x),$$

Moreover, the Cartier divisor $\mathrm{Exc}_{\mathcal{M}} \cap \widetilde{\mathcal{D}}(x)$ of $\mathrm{Exc}_{\mathcal{M}}$ is

$$\mathrm{Exc}_{\mathcal{M}} \cap \widetilde{\mathcal{D}}(x) = \begin{cases} (v=0) + (u = \text{a nonzero number in } \mathbb{F}) + (v'=0)\,, & \text{when } n = 1; \\ (v=0) + (u=0) + (v'=0) + (u'=0)\,, & \text{when } n \geqslant 2. \end{cases}$$

Proof. By Lemma 4.13.2, we have

$$\mathcal{Z}^+(x) = \mathcal{Z}_{\mathcal{N}}(x) \times_{\mathcal{N},p_+} \mathcal{M}^+ \text{ and } \mathcal{Z}^-(x) = \mathcal{Z}_{\mathcal{N}}(x') \times_{\mathcal{N},p_-} \mathcal{M}^-.$$

By Lemma 4.4.1, we have

$$\mathcal{D}^+(x) = \mathcal{N}_0(x) \times_{\mathcal{N},p_+} \mathcal{M}^+ \text{ and } \mathcal{D}^+(x) = \mathcal{N}_0(x') \times_{\mathcal{N},p_-} \mathcal{M}^-.$$

For $i = 1,2$, let $p_{?i} : \mathcal{M}_i^? \to \mathcal{N}$ be the composition $\mathcal{M}_i^? \to \mathcal{M}^? \xrightarrow{p_?} \mathcal{N}$. Let $p_{?i}^{\#} : \mathcal{O}_{\mathcal{N}} \to \mathcal{O}_{\mathcal{M}_i^?}$ be the corresponding ring homomorphism. We will prove (i) and (ii) by studying the equation of the divisor $\mathcal{Z}(x)$ on the open formal subscheme $\mathcal{M}_i^?$ for $i = 1,2$ and $? \in \{+,-\}$. For $i = 1,2$, let $t_i^+, t_i^- \in \mathcal{O}_{\mathcal{N}_0(x_0)}$ be two elements satisfying the assumption in Lemma 4.6.4.

We first prove (i). If $? = +$, the equation of the special cycle $\mathcal{Z}_{\mathcal{N}}(x)$ is $t_1^+ - \nu_x t_2^+ + p \cdot f_x$ for an element $f_x \in W[[t_1^+, t_2^+]]$ and an invertible element $\nu_x \in W[[t_1^+, t_2^+]]$. By Lemma 4.6.4, we have

$$s_+^{\#}(t_1^+ - \nu_x t_2^+) = \nu_{11}x_1 + \nu_{12}y_1^p - \nu_x\nu_{21}x_2 - \nu_x\nu_{22}y_2^p + p \cdot h,$$

where $h \in W[[t_1^+, t_2^+]]$ is an element and ν_{ij} are invertible elements in the ring $W[[x_i, y_i]]/(p + x_i y_i)$. Therefore the equation of the divisor $\mathcal{Z}(x)$ on $\mathcal{M}_i^+$ is

$$i = 1: \; p_{+1}^{\#}(t_1^+ - \nu_x t_2^+) = x_1 \cdot (\nu_{11} - \nu_x\nu_{21}v + \nu_{12}u^p v^p x_1^{p-1} - \nu_x\nu_{22}u^p x_1^{p-1} - uvx_1 \cdot h). \tag{42}$$

$$i = 2: \; p_{+2}^{\#}(t_1^+ - \nu_x t_2^+) = x_2 \cdot (\nu_{11}v' - \nu_x\nu_{21} + \nu_{12}u^p x_2^{p-1} - \nu_x\nu_{22}v'^p u^p x_2^{p-1} - uv'x_2 \cdot h). \tag{43}$$

If $? = -$, the equation of the special cycle $\mathcal{Z}_{\mathcal{N}}(x')$ is $t_1^- - \omega_{x'} t_2^- + p \cdot g_{x'}$ for an element $g_{x'} \in W[[t_1^-, t_2^-]]$ and an invertible element $\omega_{x'} \in W[[t_1^-, t_2^-]]$. By Lemma 4.6.4, we have

$$s_-^{\#}(t_1^- - \omega_{x'} t_2^-) = \omega_{11}y_1 + \omega_{12}x_1^p - \omega_{x'}\omega_{21}y_2 - \omega_{x'}\omega_{22}x_2^p + p \cdot g,$$

where $g \in W[[t_1^-, t_2^-]]$ is an element and ω_{ij} are invertible elements in the ring $W[[x_i, y_i]]/(p + x_i y_i)$. Therefore the equation of the divisor $\mathcal{Z}(x)$ on $\mathcal{M}_i^-$ is

$$(44)\quad i = 1: \; p_{-1}^{\#}(t_1^- - \omega_{x'} t_2^-) = y_1 \cdot (\omega_{11} - \omega_{x'}\omega_{21} v' + \omega_{12} u'^p v'^p y_1^{p-1} - \omega_{x'}\omega_{22} u'^p y_1^{p-1} - u'v'y_1 \cdot g).$$

$$(45)\quad i = 2: \; p_{-2}^{\#}(t_1^- - \omega_{x'} t_2^-) = y_2 \cdot (\omega_{11} v - \omega_{x'}\omega_{21} + \omega_{12} u'^p y_2^{p-1} - \omega_{x'}\omega_{22} v^p u'^p y_2^{p-1} - u'vy_2 \cdot g).$$

Define $\widetilde{\mathcal{D}}(x) := \mathcal{Z}(x) - \mathrm{Exc}_{\mathcal{M}}$. By the equations (42)-(45) of the special divisor $\mathcal{Z}(x)$, we deduce that $\widetilde{\mathcal{D}}(x)$ is an effective Cartier divisor which intersects with the exceptional divisor $\mathrm{Exc}_{\mathcal{M}}$ properly. The intersection $\widetilde{\mathcal{D}}(x) \cap \mathrm{Exc}_{\mathcal{M}} \subset \mathrm{Exc}_{\mathcal{M}}$ is given by the equation $v = \overline{\nu_x^{-1}\nu_{21}^{-1}\nu_{11}} = \overline{\omega_{x'}\omega_{21}\omega_{11}^{-1}}$, where the symbol $\overline{(\cdot)}$ means the image of the corresponding element under the map $\mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}$. Notice that $\overline{\nu_x^{-1}\nu_{21}^{-1}\nu_{11}} = \overline{\omega_{x'}\omega_{21}\omega_{11}^{-1}}$ is a nonzero number in $\mathbb{F}$. Therefore (i) is true.

Now we prove (ii). If $? = +$. By Lemma 4.4.1, the equation of the special cycle $\mathcal{D}_{\mathcal{N}}(x)$ is

$$z_x = \nu p + \left(t_1^+ - (\nu_x t_2^+)^{p^n}\right)\left(t_1^{+p^n} - \nu_x t_2^+\right) \cdot \prod_{\substack{a+b=n \\ a,b \geqslant 1}} \left(t_1^{+p^{a-1}} - (\nu_x t_2^+)^{p^{b-1}}\right)^{p-1}$$

where ν_x and ν are two invertible elements in the ring $W[[t_1^+, t_2^+]]$. By Lemma 4.6.4, we have

$$\begin{aligned} s_+^{\#}(z_x) = \nu p + &\left(\nu_{11}x_1 + \nu_{12}y_1^p - (\nu_x\nu_{21}x_2 + \nu_x\nu_{22}y_2^p)^{p^n}\right)\left((\nu_{11}x_1 + \nu_{12}y_1^p)^{p^n} - \nu_x(\nu_{21}x_2 + \nu_{22}y_2^p)\right) \\ &\cdot \prod_{\substack{a+b=n \\ a,b \geqslant 1}} \left((\nu_{11}x_1 + \nu_{12}y_1^p)^{p^{a-1}} - (\nu_x\nu_{21}x_2 + \nu_x\nu_{22}y_2^p)^{p^{b-1}}\right)^{p-1}, \end{aligned} \tag{46}$$

Therefore the equation of the divisor $\mathcal{D}_{\mathcal{N}}(x)$ on $\mathcal{M}_i^+$ is

$$i = 1: p_{+1}^{\#}(z_x) = -\nu u v x_1^2 + x_1^{m_n} \cdot \begin{cases} \tilde{\nu}_1 v + x_1^{p-1} h_1, & \text{when } n = 1; \\ h_1, & \text{when } n = 2. \end{cases} \tag{47}$$

$$i = 2: p_{+2}^{\#}(z_x) = -\nu u v' x_2^2 + x_2^{m_n} \cdot \begin{cases} \tilde{\nu}_2 v' + x_2^{p-1} h_2, & \text{when } n = 1; \\ h_2, & \text{when } n = 2. \end{cases} \tag{48}$$

Here $\nu, \tilde{\nu}_i$ are invertible elements in the ring $\mathcal{O}_{\mathcal{M}_i^+}$, h_i is an element in the ring $\mathcal{O}_{\mathcal{M}_i^+}$. By the equation (46), the integer m_n satisfies the following conditions

$$m_n \begin{cases} = 2, & \text{when } n = 1; \\ \geqslant 3, & \text{when } n \geqslant 2. \end{cases}$$

Similar method also gives the following equations of the special divisor $\mathcal{D}(x)$ on the open formal subscheme $\mathcal{M}^- = \mathcal{M}_1^- \cup \mathcal{M}_2^-$. Let $z_{x'} \in W[[t_1^-, t_2^-]]$ be the equation of the special divisor $\mathcal{Z}_{\mathcal{N}}(x')$.

$$i = 1: p_{-1}^{\#}(z_{x'}) = -\omega u' v' y_1^2 + y_1^{m_n} \cdot \begin{cases} \tilde{\omega}_1 v' + y_1^{p-1} g_1, & \text{when } n = 1; \\ g_1, & \text{when } n = 2. \end{cases} \tag{49}$$

$$i = 2: p_{-2}^{\#}(z_{x'}) = -\omega u' v y_2^2 + y_2^{m_n} \cdot \begin{cases} \tilde{\omega}_2 v + y_2^{p-1} g_2, & \text{when } n = 1; \\ g_2, & \text{when } n = 2. \end{cases} \tag{50}$$

Here $\omega, \tilde{\omega}_i$ are invertible elements in the ring $\mathcal{O}_{\mathcal{M}_i^-}$, g_i is an element in the ring $\mathcal{O}_{\mathcal{M}_i^-}$.

Define $\widetilde{\mathcal{D}}(x) := \mathcal{D}(x) - 2\cdot \mathrm{Exc}_{\mathcal{M}}$. By the equations (47)-(50) of the difference divisor $\mathcal{D}(x)$, we deduce that $\widetilde{\mathcal{D}}(x) := \mathcal{D}(x) - 2\cdot \mathrm{Exc}_{\mathcal{M}}$ is an effective Cartier divisor which intersects with the exceptional divisor properly. The intersection $\widetilde{\mathcal{D}}(x)\cap \mathrm{Exc}_{\mathcal{M}}$ is an effective Cartier divisor on the exceptional divisor $\mathrm{Exc}_{\mathcal{M}}$. More specifically,

$$\widetilde{\mathcal{D}}(x)\cap \mathrm{Exc}_{\mathcal{M}} = \begin{cases} (v=0)+(u=\text{a nonzero number in }\mathbb{F})+(v'=0)\,, & \text{when } n=1;\\ (v=0)+(u=0)+(v'=0)+(u'=0)\,, & \text{when } n\geqslant 2.\end{cases}$$

Therefore (ii) is true. □

Corollary 5.3.2. *Let $x\in\mathbb{B}$ be a nonzero element such that $q(x)\in\mathbb{Z}_p$ such that $n:=\nu_p(q(x))\geqslant 0$. Define $\widetilde{\mathcal{Z}}(x)\subset\mathcal{M}$ to be the strict transformation of the cycle $\mathcal{Z}_{\mathcal{N}(x_0)}(\{x\})\subset\mathcal{N}(x_0)$ under the blow up morphism $\pi:\mathcal{M}\to\mathcal{N}(x_0)$. We have the following identity of Cartier divisor on $\mathcal{M}$:*

$$\widetilde{\mathcal{Z}}(x) = \sum_{i=0}^{[n/2]}\widetilde{\mathcal{D}}(p^{-i}x).$$

(a) *The following identities of Cartier divisors on $\mathcal{M}$ hold,*

$$\mathcal{Z}(x) = (n+1)\cdot \mathrm{Exc}_{\mathcal{M}} + \widetilde{\mathcal{Z}}(x).$$

(b) *The following identities in the group $\mathrm{Gr}^2 K_0^{\mathrm{Exc}_{\mathcal{M}}}(\mathcal{M})\simeq \mathrm{Pic}(\mathrm{Exc}_{\mathcal{M}})$ holds,*

$$[\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\widetilde{\mathcal{D}}(x)}] = [\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}\otimes_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\widetilde{\mathcal{D}}(x)}] = [\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}\cap\widetilde{\mathcal{D}}(x)}] = \begin{cases}\mathcal{O}(1,0), & \text{when } n=0;\\ \mathcal{O}(2,1), & \text{when } n=1;\\ \mathcal{O}(2,2), & \text{when } n\geqslant 2.\end{cases} \tag{51}$$

$$[\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\mathcal{Z}(x)}] = \mathcal{O}(0,-1). \tag{52}$$

Proof. By the definition of difference divisors in Definition 4.13.4, we have the following equality as effective Cartier divisors on $\mathcal{M}$,

$$\mathcal{Z}(x) = \sum_{i=0}^{[n/2]}\mathcal{D}(p^{-i}x).$$

By Proposition 5.3.1, we have

$$\mathcal{Z}(x) = (n+1)\cdot\mathrm{Exc}_{\mathcal{M}} + \sum_{i=0}^{[n/2]}\widetilde{\mathcal{D}}(p^{-i}x),$$

and the effective Cartier divisor $\sum_{i=0}^{[n/2]}\widetilde{\mathcal{D}}(p^{-i}x)$ intersects with the exceptional divisor $\mathrm{Exc}_{\mathcal{M}}$ properly. Therefore we conclude that

$$\widetilde{\mathcal{Z}}(x) = \sum_{i=0}^{[n/2]}\widetilde{\mathcal{D}}(p^{-i}x).$$

Hence $\mathcal{Z}(x) = (n+1)\cdot\mathrm{Exc}_{\mathcal{M}} + \widetilde{\mathcal{Z}}(x)$.

The formula (51) in (b) follows from Proposition 5.3.1. For the formula (52) in (b): By Lemma 5.1.1, we know that $\mathcal{O}_{\mathcal{Z}(x)} = (n+1)\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}} + \sum_{i=0}^{[n/2]}\mathcal{O}_{\widetilde{\mathcal{D}}(p^{-i}x)}$ in $\mathrm{Gr}^1 K_0^{\mathcal{Z}(x)}(\mathcal{M})$. Therefore by Lemma

5.2.1 and (51), we have the following equality in $\mathrm{Gr}^2 K_0^{\mathrm{Exc}_{\mathcal{M}}}(\mathcal{M}) \simeq \mathrm{Pic}(\mathrm{Exc}_{\mathcal{M}})$,

$$[\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(x)}] = (n+1)[\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}] + \sum_{i=0}^{[n/2]} [\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\widetilde{\mathcal{D}}(p^{-i}x)}]$$
$$= (n+1)\mathcal{O}(-1,-1) + \mathcal{O}(n+1,n) = \mathcal{O}(0,-1).$$

□

5.4. **Local Hecke correspondences and the exceptional divisor.** Recall that for an element $x \in \mathbb{B}$ such that $\nu_p(q(x)) \geqslant 1$, we defined closed formal subschemes $\mathcal{N}_0^{\mathrm{I}+}(x), \mathcal{N}_0^{\mathrm{I}-}(x), \mathcal{N}_0^{\mathrm{II}+}(x), \mathcal{N}_0^{\mathrm{II}-}(x)$ of $\mathcal{M}$.

Lemma 5.4.1. *Let $x \in \mathbb{B}$ be an element such that $\nu_p(q(x)) \geqslant 0$. The formal subschemes $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x)$, $\mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x})$, $\mathcal{N}_0^{\mathrm{II}+}(x)$ and $\mathcal{N}_0^{\mathrm{II}-}(x')$ (the later two spaces are only defined for elements x such that $\nu_p(q(x)) \geqslant 1$) are all closed formal subschemes of the special divisor $\mathcal{Z}(x)$.*

Proof. By Lemma 4.9.2, the morphisms $\mathrm{st}^{\mathrm{I}+}_{x_0\cdot x}$, $\mathrm{st}^{\mathrm{I}-}_{x_0\cdot \overline{x}}$, $\mathrm{st}^{\mathrm{II}+}_{x}$ and $\mathrm{st}^{\mathrm{II}-}_{x'}$ sends the corresponding source formal schemes to $\mathcal{Z}_{\mathcal{N}(x_0)}(x)$. Therefore the strict transforms of the corresponding formal schemes $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x), \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x}), \mathcal{N}_0^{\mathrm{II}+}(x)$ and $\mathcal{N}_0^{\mathrm{II}-}(x')$ are all mapped into the direct base change $\mathcal{Z}_{\mathcal{N}(x_0)}(x) \times_{\mathcal{N}(x_0)} \mathcal{M} = \mathcal{Z}(x)$. □

Remark 5.4.2. By Lemma 4.9.3, we know that

- The regular divisor $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) = \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x})$ if $\nu_p(q(x)) = 0$.
- The regular divisor $\mathcal{N}_0^{\mathrm{II}+}(x) = \mathcal{N}_0^{\mathrm{II}-}(x')$ if $\nu_p(q(x)) = 1$.

Notice that over the closed formal subschemes $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x)$, $\mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x})$, $\mathcal{N}_0^{\mathrm{II}+}(x)$ and $\mathcal{N}_0^{\mathrm{II}-}(x')$, the quasi-isogenies $x_0 \cdot x$ and $x_0 \cdot \overline{x}$ lift to isogenies by Lemma 5.4.1. We still use $x_0 \cdot x$ and $x_0 \cdot \overline{x}$ to denote these two isogenies.

Lemma 5.4.3. *Let $x \in \mathbb{B}$ be an element such that $\nu_p(q(x)) \geqslant 1$.*

(I): + *Over the formal scheme $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x)$,*

$$x_0 \cdot x = \text{a cyclic isogeny}, \qquad x = \text{a cyclic isogeny},$$
$$x_0 \cdot \overline{x} = p \times \text{a cyclic isogeny}, \qquad x' = \text{a cyclic isogeny}.$$

− *Over the formal scheme $\mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x})$,*

$$x_0 \cdot x = p \times \text{a cyclic isogeny}, \qquad x = \text{a cyclic isogeny},$$
$$x_0 \cdot \overline{x} = \text{a cyclic isogeny}, \qquad x' = \text{a cyclic isogeny}.$$

(II): + *Over the formal scheme $\mathcal{N}_0^{\mathrm{II}+}(x)$,*

$$x_0 \cdot x = p \times \text{a cyclic isogeny}, \qquad x_0 \cdot \overline{x} = p \times \text{a cyclic isogeny}.$$

When $\nu_p(q(x)) \geqslant 2$, we also have

$$x = \text{a cyclic isogeny}, \qquad x' = p \times \text{a cyclic isogeny}.$$

– *Over the formal scheme* $\mathcal{N}_0^{\mathrm{II}-}(x')$,

$$x_0 \cdot x = p \times \text{a cyclic isogeny}, \qquad x_0 \cdot \overline{x} = p \times \text{a cyclic isogeny}.$$

When $\nu_p(q(x)) \geqslant 2$, *we also have*

$$x = p \times \text{a cyclic isogeny}, \qquad x' = \text{a cyclic isogeny}.$$

Proof. Let $n = \nu_p(q(x))$. Let S be a W-scheme such that p is locally nilpotent. We first consider the case (I)+. Let

$$\left(X_1 \xrightarrow{(x_0)_1} X_1', (\rho_1, \rho_1')\right), \quad \left(X_2 \xrightarrow{(x_0)_2} X_2', (\rho_2, \rho_2')\right) \tag{53}$$

be an object in the set $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x)(S)$. Since the formal scheme $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x)$ is contained $\mathcal{N}_0(x_0 \cdot x) \times_{\mathrm{st}^{\mathrm{I}+}_{x_0\cdot x}, \mathcal{N}(x_0)} \mathcal{M}$, the quasi-isogeny $x_0 \cdot x$ lifts to a cyclic isogeny $\pi : X_1 \to X_2'$.

The cyclic isogeny π factorizes into a composition of n degree p isogenies $\pi = \pi_n \circ \cdots \circ \pi_1$. The isogeny π_1 is isomorphic to $(x_0)_1$, while π_n is isomorphic to $(x_0)_2$. Hence x lifts to $\pi_{n-1} \circ \cdots \circ \pi_1$. Then $\overline{x}$ lifts to $\pi_1^\vee \circ \cdot \circ \pi_{n-1}^\vee$. Therefore

$$x_0 \cdot \overline{x} = (x_0)_1 \circ \pi_1^\vee \circ \cdots \circ \pi_{n-1}^\vee = \pi_1 \circ \pi_1^\vee \circ \cdots \circ \pi_{n-1}^\vee = q(x_0) \times \pi_2^\vee \circ \cdots \circ \pi_{n-1}^\vee = p \times \text{a cyclic isogeny}.$$

$$x = x_0^{-1} \circ x_0 \cdot x = ((x_0)_2)^{-1} \circ \pi_n \circ \cdots \circ \pi_1 = \pi_{n-1} \circ \cdots \circ \pi_1 = \text{a cyclic isogeny}.$$

$$x' = (x_0)_2 \circ x \circ ((x_0)_1)^{-1} = \pi_n \circ \cdots \circ \pi_2 = \text{a cyclic isogeny}.$$

Therefore we have shown that $x_0 \cdot x$ is a cyclic isogeny, while $x_0 \cdot \overline{x}$, x and x' are of the form $p \times$ a cyclic isogeny over the formal scheme $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x)$. The proof for the other cases are similar so we omit it. □

Let $x \in \mathbb{B}$ be an element such that $\nu_p(q(x)) \geqslant 0$. We know that the formal schemes $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x)$, $\mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x})$, $\mathcal{N}_0^{\mathrm{II}+}(x)$ and $\mathcal{N}_0^{\mathrm{II}-}(x')$ (the later two spaces are only defined for elements x such that $\nu_p(q(x)) \geqslant 1$) are isomorphic to the blow-up along the unique closed point of the corresponding cyclic deformations spaces $\mathcal{N}_0(x_0 \cdot x)$, $\mathcal{N}_0(x_0 \cdot \overline{x})$, $\mathcal{N}_0(x)$ and $\mathcal{N}_0(x')$. Denote by $\mathrm{Exc}^{\mathrm{I}+}_{x_0\cdot x}$, $\mathrm{Exc}^{\mathrm{I}-}_{x_0\cdot \overline{x}}$, $\mathrm{Exc}^{\mathrm{II}+}_{x}$ and $\mathrm{Exc}^{\mathrm{II}-}_{x'}$ the corresponding exceptional divisors. They are all isomorphic to $\mathbb{P}^1_{\mathbb{F}}$. By the definition of these formal schemes,

$$\mathrm{Exc}^{\mathrm{I}+}_{x_0\cdot x} = \mathrm{Exc}_{\mathcal{M}} \cap \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x), \quad \mathrm{Exc}^{\mathrm{I}-}_{x_0\cdot \overline{x}} = \mathrm{Exc}_{\mathcal{M}} \cap \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x}),$$
$$\mathrm{Exc}^{\mathrm{II}+}_{x} = \mathrm{Exc}_{\mathcal{M}} \cap \mathcal{N}_0^{\mathrm{II}+}(x), \quad \mathrm{Exc}^{\mathrm{II}-}_{x'} = \mathrm{Exc}_{\mathcal{M}} \cap \mathcal{N}_0^{\mathrm{II}-}(x').$$

Hence $\mathrm{Exc}^{\mathrm{I}+}_{x_0\cdot x}$, $\mathrm{Exc}^{\mathrm{I}-}_{x_0\cdot \overline{x}}$, $\mathrm{Exc}^{\mathrm{II}+}_{x}$ and $\mathrm{Exc}^{\mathrm{II}-}_{x'}$ are Cartier divisors on $\mathrm{Exc}_{\mathcal{M}}$.

Proposition 5.4.4. *Let* $x \in \mathbb{B}$ *be an element such that* $n := \nu_p(q(x)) \geqslant 0$.

(I) *The Cartier divisors* $\mathrm{Exc}^{\mathrm{I}+}_{x_0\cdot x}$ *and* $\mathrm{Exc}^{\mathrm{I}-}_{x_0\cdot \overline{x}}$ *of* $\mathrm{Exc}_{\mathcal{M}}$ *are*

$$\mathrm{Exc}^{\mathrm{I}+}_{x_0\cdot x} = \begin{cases} (v = \text{a nonzero number in } \mathbb{F}), & \text{when } n = 0; \\ (v = 0), & \text{when } n \geqslant 1. \end{cases}$$

$$\mathrm{Exc}^{\mathrm{I}-}_{x_0\cdot \overline{x}} = \begin{cases} (v' = \text{a nonzero number in } \mathbb{F}), & \text{when } n = 0; \\ (v' = 0), & \text{when } n \geqslant 1. \end{cases}$$

(II) *When* $n \geqslant 1$, *the Cartier divisors* $\mathrm{Exc}_x^{\mathrm{II}+}$ *and* $\mathrm{Exc}_{x'}^{\mathrm{II}-}$ *of* $\mathrm{Exc}_{\mathcal{M}}$ *are*

$$\mathrm{Exc}_x^{\mathrm{II}+} = \begin{cases} (u = \text{a nonzero number in } \mathbb{F}), & \text{when } n = 1; \\ (u = 0), & \text{when } n \geqslant 2. \end{cases}$$

$$\mathrm{Exc}_{x'}^{\mathrm{II}-} = \begin{cases} (u' = \text{a nonzero number in } \mathbb{F}), & \text{when } n = 1; \\ (u' = 0), & \text{when } n \geqslant 2. \end{cases}$$

Proof. We first consider the case (I+). When $n = 0$, by Lemma 5.4.1, we have $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \subset \mathcal{Z}(x)$. By Proposition 5.3.1, we know that $\mathcal{Z}(x) = \mathrm{Exc}_{\mathcal{M}} + \widetilde{\mathcal{D}}(x)$. Then $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \subset \widetilde{\mathcal{D}}(x)$ because $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x)$ intersects with $\mathrm{Exc}_{\mathcal{M}}$ properly. Hence

$$\mathbb{P}^1_{\mathbb{F}} \simeq \mathrm{Exc}^{\mathrm{I}+}_{x_0 \cdot x} = \mathrm{Exc}_{\mathcal{M}} \cap \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \subset \mathrm{Exc}_{\mathcal{M}} \cap \widetilde{\mathcal{D}}(x).$$

By Proposition 5.3.1, the intersection $\mathrm{Exc}_{\mathcal{M}} \cap \widetilde{\mathcal{D}}(x)$ is given by the equation $v =$ a nonzero number in $\mathbb{F}$, which also cuts out a projective line $\mathbb{P}^1_{\mathbb{F}}$ in $\mathrm{Exc}_{\mathcal{M}}$. Hence the Cartier divisor $\mathrm{Exc}^{\mathrm{I}+}_{x_0 \cdot x}$ is given by the equation ($v =$ a nonzero number in $\mathbb{F}$) in $\mathrm{Exc}_{\mathcal{M}}$. Notice that $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) = \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x})$ by Remark 5.4.2. Hence $\mathrm{Exc}^{\mathrm{I}-}_{x_0 \cdot \overline{x}} = \mathrm{Exc}^{\mathrm{I}+}_{x_0 \cdot x}$. Therefore the Cartier divisor $\mathrm{Exc}^{\mathrm{I}-}_{x_0 \cdot \overline{x}}$ is given by the equation ($v' =$ a nonzero number in $\mathbb{F}$) since $v' \cdot v = 1$.

Now we consider the case $n \geqslant 1$. Let's assume that $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \cap (\mathcal{M}_2^+ \cup \mathcal{M}_1^-) \neq \varnothing$. Let $\mathcal{M}_{v'}$ be the closed formal subscheme of $\mathcal{M}$ cut out by the equation $v' = 0$. Notice that $v' = 0$ also implies that $x_1 = y_2 = 0$. Therefore

$$\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \cap \mathcal{M}_{v'} \subset \left(\mathrm{st}^{\mathrm{I}+}_{x_0 \cdot x}\left(\mathcal{N}_0(x_0 \cdot x)\right) \cap (x_1 = y_2 = 0)\right) \times_{\mathcal{N}(x_0)} \mathcal{M}.$$

By our convention in Remark 4.6.3, $x_1 = y_2 = 0$ cuts out the closed formal subscheme $\mathcal{N}(x_0)^{\mathrm{VF}}$ of $\mathcal{N}(x_0)$. Therefore

$$\mathrm{st}^{\mathrm{I}+}_{x_0 \cdot x}\left(\mathcal{N}_0(x_0 \cdot x)\right) \cap (x_1 = y_2 = 0) = \mathcal{N}_0(x_0 \cdot x) \times_{\mathrm{st}^{\mathrm{I}+}_{x_0 \cdot x}, \mathcal{N}(x_0)} \mathcal{N}(x_0)^{\mathrm{VF}} \simeq \operatorname{Spec} \mathbb{F}$$

by Lemma 4.8.2 and Remark 4.8.3. Hence $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \cap \mathcal{M}_{v'} \subset \operatorname{Spec} \mathbb{F} \times_{\mathcal{N}(x_0)} \mathcal{M} = \mathrm{Exc}_{\mathcal{M}}$.

We assume that $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \cap \mathcal{M}_2^+ \neq \varnothing$ (the argument for the case $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \cap \mathcal{M}_1^- \neq \varnothing$ is similar). Let $f \in \mathcal{O}_{\mathcal{M}_2^+}$ be the equation of the regular divisor $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x)$ in $\mathcal{M}_2^+$. The inclusion $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \cap \mathcal{M}_{v'} \subset \mathrm{Exc}_{\mathcal{M}}$ implies that

$$(x_2) \subset (f, v') \subset \mathcal{O}_{\mathcal{M}_2^+} \simeq W[v', u][[x_2]]/(p + uv'x_2^2).$$

Therefore there exist $a, b \in \mathcal{O}_{\mathcal{M}_2^+}$ such that $x_2 = av' + bf$.

Claim: the element a is invertible.

Proof of the claim: Let $\overline{a} := a \bmod (x_2)$ be an element in $\mathbb{F}[v', u]$. Then $\overline{a} \neq 0$ because otherwise $x_2 | f$, which is impossible. Then the intersection $\mathrm{Exc}^{\mathrm{I}+}_{x_0 \cdot x} \cap \mathcal{M}_2^+$ is given by $(\overline{a} \cdot v' = 0) = (\overline{a} = 0) + (v' = 0)$ by the equation $x_2 = av' + bf$. However, the intersection $\mathrm{Exc}^{\mathrm{I}+}_{x_0 \cdot x} \cap \mathcal{M}_2^+$ is an open subvariety of $\mathbb{P}^1_{\mathbb{F}}$. Therefore we must have $(\overline{a} = 0) = \varnothing$ and $\mathrm{Exc}^{\mathrm{I}+}_{x_0 \cdot x} = (v' = 0)$. Hence $\overline{a}$ is invertible, which implies that a is invertible.

The invertibility of a implies that the element $bf = x_2 - av'$ is a regular element because the quotient ring $\mathcal{O}_{\mathcal{M}_2^+}/(x_2 - av')$ is regular. Therefore the element b must be invertible because f is not invertible. Therefore we conclude that $f = \tilde{a}v' + \tilde{b}x_2$ for invertible elements $\tilde{a}, \tilde{b} \in \mathcal{O}_{\mathcal{M}_2^+}$. Then

$$\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \cap \mathcal{M}_2^+ \simeq \operatorname{Spf} W[u][[x_2]]/(p - \tilde{a}^{-1}\tilde{b}ux_2^3).$$

By the above equation, the multiplicity of the exceptional divisor $\mathrm{Exc}_{x_0\cdot x}^{\mathrm{I}+}$ in $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x)_{\mathbb{F}} = \mathrm{div}(p)$ of $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x)$ is 3. Notice that by Lemma 4.5.1, this multiplicity is $r(n)$, but $r(n) \neq 3$ when $p > 2$, this is a contradiction. Therefore the assumption $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \cap (\mathcal{M}_2^+ \cup \mathcal{M}_1^-) \neq \varnothing$ is wrong. Since $\mathrm{Exc}_{x_0\cdot x}^{\mathrm{I}+} \subset \widetilde{\mathcal{Z}}(x) \cap \mathrm{Exc}_{\mathcal{M}}$ and the latter is a summation of divisors $(v = 0)$, $(v' = 0)$, $(u = 0)$, $(u' = 0)$ and (u = a nonzero number in $\mathbb{F}$) on $\mathrm{Exc}_{\mathcal{M}}$. Therefore the only possibility is

$$\mathrm{Exc}_{x_0\cdot x}^{\mathrm{I}+} = (v = 0).$$

The proof for the other cases are similar, so we omit it. □

Remark 5.4.5. In the proof of Proposition 5.4.4, the assumption $p > 2$ is used solely to deduce $r(n) \neq 3$ and reach a contradiction; we expect a proof avoiding this. This is the only place on the geometric part where $p > 2$ is needed.

5.5. Decomposition of the difference divisor.

Lemma 5.5.1. *Let $x \in \mathbb{B}$ be an element such that $\nu_p(q(x)) \geqslant 2$. Then the regular divisors $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x), \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x}), \mathcal{N}_0^{\mathrm{II}+}(x), \mathcal{N}_0^{\mathrm{II}-}(x')$ are all contained in the divisor $\widetilde{\mathcal{D}}(x)$.*

Proof. By the moduli interpretation of the divisors $\mathcal{D}^+(x)$ and $\mathcal{D}^-(x)$ in Remark 4.13.5 and Lemma 5.4.3, we have

$$\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) = \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \cap \mathcal{M}^+ \bigcup \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \cap \mathcal{M}^- \subset \mathcal{D}^+(x) \bigcup \mathcal{D}^-(x) = \mathcal{D}(x),$$
$$\mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x}) = \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x}) \cap \mathcal{M}^+ \bigcup \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x}) \cap \mathcal{M}^- \subset \mathcal{D}^+(x) \bigcup \mathcal{D}^-(x) = \mathcal{D}(x).$$

Therefore the two divisors $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x), \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x}) \subset \widetilde{\mathcal{D}}(x)$ since they intersect the exceptional divisor $\mathrm{Exc}_{\mathcal{M}}$ properly.

By Proposition 5.4.4, we know that

$$\mathrm{Exc}_x^{\mathrm{II}+} = (u = 0), \quad \mathrm{Exc}_{x'}^{\mathrm{II}-} = (u' = 0).$$

Hence $\mathcal{N}_0^{\mathrm{II}+}(x) \subset \mathcal{M}^+$ and $\mathcal{N}_0^{\mathrm{II}-}(x') \subset \mathcal{M}^-$. Therefore by Lemma 5.4.3

$$\mathcal{N}_0^{\mathrm{II}+}(x) = \mathcal{N}_0^{\mathrm{II}+}(x) \cap \mathcal{M}^+ \subset \mathcal{D}^+(x) \subset \mathcal{D}(x),$$
$$\mathcal{N}_0^{\mathrm{II}-}(x') = \mathcal{N}_0^{\mathrm{II}-}(x') \cap \mathcal{M}^- \subset \mathcal{D}^-(x) \subset \mathcal{D}(x).$$

Therefore the two divisors $\mathcal{N}_0^{\mathrm{II}+}(x), \mathcal{N}_0^{\mathrm{II}-}(x') \subset \widetilde{\mathcal{D}}(x)$ since they intersect the exceptional divisor $\mathrm{Exc}_{\mathcal{M}}$ properly. □

Lemma 5.5.2. *Let $x \in \mathbb{B}$ be a non-zero element such that $\nu_p(q(x)) \geqslant 0$. We have the following decomposition of the effective Cartier divisor $\widetilde{\mathcal{D}}(x)$,*

$$\widetilde{\mathcal{D}}(x) = \begin{cases} \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x), & \text{if } \nu_p(q(x)) = 0; \\ \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) + \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x}) + \mathcal{N}_0^{\mathrm{II}+}(x), & \text{if } \nu_p(q(x)) = 1; \\ \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) + \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x}) + \mathcal{N}_0^{\mathrm{II}+}(x) + \mathcal{N}_0^{\mathrm{II}-}(x'), & \text{if } \nu_p(q(x)) \geqslant 2. \end{cases}$$

Proof. Let $n = \nu_p(q(x))$. Denote by $H(x)$ the effective Cartier divisor on the right hand side, i.e.,

$$H(x) = \begin{cases} \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x), & \text{if } \nu_p(q(x)) = 0; \\ \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) + \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x}) + \mathcal{N}_0^{\mathrm{II}+}(x), & \text{if } \nu_p(q(x)) = 1; \\ \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) + \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x}) + \mathcal{N}_0^{\mathrm{II}+}(x) + \mathcal{N}_0^{\mathrm{II}-}(x'), & \text{if } \nu_p(q(x)) \geqslant 2. \end{cases}$$

Let $R(x) = \widetilde{\mathcal{D}}(x) - H(x)$. By Proposition 5.3.1 and Proposition 5.4.4, we know that

$$\widetilde{\mathcal{D}}(x) \cap \mathrm{Exc}_{\mathcal{M}} = H(x) \cap \mathrm{Exc}_{\mathcal{M}}$$

as Cartier divisors on $\mathrm{Exc}_{\mathcal{M}}$.

If $R(x) \neq \varnothing$, it must be an effective Cartier divisor by Lemma 5.5.1, and intersects with $\mathrm{Exc}_{\mathcal{M}}$ properly. The identity $\widetilde{\mathcal{D}}(x) \cap \mathrm{Exc}_{\mathcal{M}} = H(x) \cap \mathrm{Exc}_{\mathcal{M}}$ as Cartier divisors on $\mathrm{Exc}_{\mathcal{M}}$ implies that $R(x) \cap \mathrm{Exc}_{\mathcal{M}} = \varnothing$. However, the exceptional divisor is the reduced locus of the formal scheme $\mathcal{M}$. We conclude that $R(x) = \varnothing$. Therefore $\widetilde{\mathcal{D}}(x) = H(x)$. □

Remark 5.5.3. We can also consider the difference divisor associated to $\mathcal{Y}$-cycles. Let $x \in \mathbb{B}$ be an element such that $\nu_p(q(x)) \geqslant -1$. Define $\mathcal{D}^{\mathcal{Y}}(x) = \mathcal{Y}(x) - \mathcal{Y}(p^{-1}x)$. By the identity $\mathcal{Y}(x) = (\iota^{\mathcal{M}})^*\mathcal{Z}(x_0 \cdot x)$, we have $\mathcal{D}^{\mathcal{Y}}(x) = (\iota^{\mathcal{M}})^*\mathcal{D}(x_0 x)$. Define $\widetilde{\mathcal{D}}^{\mathcal{Y}}(x) = (\iota^{\mathcal{M}})^*\widetilde{\mathcal{D}}(x_0 \cdot x)$. Then by Lemma 5.5.2 and (37), we have

$$\widetilde{\mathcal{D}}^{\mathcal{Y}}(x) = \begin{cases} \mathcal{N}_0^{\mathrm{II}+}(px), & \text{if } \nu_p(q(x)) = -1; \\ \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) + \mathcal{N}_0^{\mathrm{II}+}(px) + \mathcal{N}_0^{\mathrm{II}-}(px'), & \text{if } \nu_p(q(x)) = 0; \\ \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) + \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{x}) + \mathcal{N}_0^{\mathrm{II}+}(px) + \mathcal{N}_0^{\mathrm{II}-}(px'), & \text{if } \nu_p(q(x)) \geqslant 1. \end{cases} \tag{54}$$

5.6. Derived special cycles ${}^{\mathbb{L}}\mathcal{Z}(L)$.

Lemma 5.6.1. *Let $x, y \in \mathbb{B}$ be two linearly independent elements. Then*

(a) *The two effective Cartier divisors $\widetilde{\mathcal{Z}}(x)$ and $\mathcal{Z}(y)$ intersect properly.*
(b) *The irreducible components of the intersection $\widetilde{\mathcal{Z}}(x) \cap \mathcal{Z}(y)$ are of the form*

$$\mathrm{Spf}\, W_s \ \text{ or } \ \mathbb{P}^1_{\mathbb{F}} \subset \mathrm{Exc}_{\mathcal{M}}.$$

Here W_s is the ring of definition of a quasi-canonical lifting of level s.

Proof. By Lemma 5.5.2, an irreducible component of the divisor $\widetilde{\mathcal{Z}}(x)$ are of the form $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot \tilde{x}), \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{\tilde{x}}), \mathcal{N}_0^{\mathrm{II}+}(\tilde{x}), \mathcal{N}_0^{\mathrm{II}-}(\tilde{x}')$ where $\tilde{x} = p^{-i}x$ for some positive integer i. It's sufficient to prove (a) and (b) for the intersections of the divisors $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot \tilde{x}), \mathcal{N}_0^{\mathrm{I}-}(x_0 \cdot \overline{\tilde{x}}), \mathcal{N}_0^{\mathrm{II}+}(\tilde{x}), \mathcal{N}_0^{\mathrm{II}-}(\tilde{x}')$ and the divisor $\mathcal{Z}(y)$. Without loss of generality, we can assume $\tilde{x} = x$.

We first consider the intersection $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \cap \mathcal{Z}(y)$. Denote by $\pi^{\mathrm{I}+} : \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \to \mathcal{N}_0(x_0 \cdot x)$ the blow up morphism, then

$$\begin{aligned} \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \cap \mathcal{Z}(y) &\subset (\mathcal{N}_0(x_0 \cdot x) \cap \mathcal{Z}_{\mathcal{N}}(x_0 \cdot y)) \times_{\mathcal{N}_0(x_0 \cdot x), \pi^{\mathrm{I}+}} \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \\ &\subset (\mathcal{Z}_{\mathcal{N}}(x_0 \cdot x) \cap \mathcal{Z}_{\mathcal{N}}(x_0 \cdot y)) \times_{\mathcal{N}_0(x_0 \cdot x), \pi^{\mathrm{I}+}} \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x). \end{aligned}$$

By Lemma 4.3.2, we know that $\dim \mathcal{Z}_{\mathcal{N}}(x_0 \cdot x) \cap \mathcal{Z}_{\mathcal{N}}(x_0 \cdot y) = 1$, hence $\dim \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) \cap \mathcal{Z}(y) \leqslant \dim\, (\mathcal{Z}_{\mathcal{N}}(x_0 \cdot x) \cap \mathcal{Z}_{\mathcal{N}}(x_0 \cdot y)) \times_{\mathcal{N}_0(x_0 \cdot x)} \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x) = 1$. Therefore $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x)$ and $\mathcal{Z}(y)$ intersect properly. Part (b) follows from Lemma 4.3.2 and Lemma 4.5.1. The proof for the other intersections are similar so we omit it. □

For a lattice $M \subset \mathbb{B}$, define $\min(M) = \min_{x\in M}\{\nu_p(q(x))\}$. We say an element $x \in M$ is a minimal element of M if $\nu_p(q(x)) = \min(M)$. It's easy to see that if x is a minimal element of M, we have $p^{-1}x \notin M$.

Proposition 5.6.2. *Let $M \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 2. Let $x \in M$ be a minimal element of M. Let $y \in M$ be another element such that $\{x, y\}$ is a $\mathbb{Z}_p$-basis of M. The element $[\mathcal{O}_{\tilde{\mathcal{Z}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)}] \in \mathrm{Gr}^2 K_0^{\mathcal{Z}(M)}(\mathcal{M})$ is independent of the choices of the minimal element x and the element y.*

Proof. Let $n = \min(M)$. If $n < 0$, then the element $\mathcal{O}_{\tilde{\mathcal{Z}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)} = 0$ in the group $K_0^{\mathcal{Z}(M)}(\mathcal{M})$ for all choices of minimal elements $x \in M$ and $y \in M$ such that $\{x, y\}$ is a basis of M. Therefore we only need to consider the case $n \geqslant 0$.

Let $x' \in M$ be another minimal element of M, i.e., $\nu_p(q(x')) = n$. Let $y' \in M$ be another element such that $\{x', y'\}$ is a basis of M. Then there exist $a, b, c, d \in \mathbb{Z}_p$ such that

$$x' = ax + by, \quad y' = cx + dy, \quad ad - bc \in \mathbb{Z}_p^\times.$$

We first consider some special cases.

Case 1 : $x' = x$, $y' = cx + y$. Then by Lemma 5.6.1, we have

$$\mathcal{O}_{\tilde{\mathcal{Z}}(x')} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y')} = \mathcal{O}_{\tilde{\mathcal{Z}}(x)} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y')} = \mathcal{O}_{\tilde{\mathcal{Z}}(x)\cap\mathcal{Z}(y')}.$$

By the moduli interpretations of the special cycles, we have $\tilde{\mathcal{Z}}(x) \cap \mathcal{Z}(y') = \tilde{\mathcal{Z}}(x) \cap \mathcal{Z}(y)$. Therefore

$$\mathcal{O}_{\tilde{\mathcal{Z}}(x')} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y')} = \mathcal{O}_{\tilde{\mathcal{Z}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)}.$$

Case 2 : $x' = x + ay$, $y' = y$. Let $m = \nu_p(q(y))$. We have:

$$\begin{aligned}
&[\mathcal{O}_{\tilde{\mathcal{Z}}(x')} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y')}] = [\mathcal{O}_{\tilde{\mathcal{Z}}(x')} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)}] = [\mathcal{O}_{\tilde{\mathcal{Z}}(x')} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\tilde{\mathcal{Z}}(y)}] + (m+1)\cdot[\mathcal{O}_{\tilde{\mathcal{Z}}(x')} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}] \\
=&[\mathcal{O}_{\mathcal{Z}(x')} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\tilde{\mathcal{Z}}(y)}] - (n+1)\cdot[\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\tilde{\mathcal{Z}}(y)}] + (m+1)\cdot\mathcal{O}(n+1, n) \\
=&[\mathcal{O}_{\mathcal{Z}(x')\cap\tilde{\mathcal{Z}}(y)}] - (n+1)\cdot\mathcal{O}(m+1, m) + (m+1)\cdot\mathcal{O}(n+1, n) = [\mathcal{O}_{\mathcal{Z}(x)\cap\tilde{\mathcal{Z}}(y)}] + \mathcal{O}(0, n-m).
\end{aligned}$$

On the other hand, similar computations also apply to $[\mathcal{O}_{\tilde{\mathcal{Z}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)}]$ just by replacing x' by x in the above computations. We conclude that

$$[\mathcal{O}_{\tilde{\mathcal{Z}}(x')} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y')}] = [\mathcal{O}_{\tilde{\mathcal{Z}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)}] = [\mathcal{O}_{\mathcal{Z}(x)\cap\tilde{\mathcal{Z}}(y)}] + \mathcal{O}(0, n-m) \text{ in } \mathrm{Gr}^2 K_0^{\mathcal{Z}(M)}(\mathcal{M}).$$

Now we come back to prove the proposition. There are two situations.

- If $a \in \mathbb{Z}_p^\times$. Scaling x' by $a^{-1} \in \mathbb{Z}_p^\times$, we can assume $a = 1$. Then $y' = cx' + (d - bc)y$. Scaling y' by $(d - bc)^{-1} \in \mathbb{Z}_p^\times$, we can assume $d - bc = 1$. Then $x' = x + by, y' = cx' + y$. Therefore
$$[\mathcal{O}_{\tilde{\mathcal{Z}}(x')} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y')}] \overset{\text{Case 1}}{=} [\mathcal{O}_{\tilde{\mathcal{Z}}(x')} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)}] \overset{\text{Case 2}}{=} [\mathcal{O}_{\tilde{\mathcal{Z}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)}].$$
- If $\nu_p(a) \geqslant 1$. Then $b, c \in \mathbb{Z}_p^\times$. Scaling x' by b^{-1}, we can assume $b = 1$. Then $y' = dx' + (c - ad)x$. Scaling y' by $(c - ad)^{-1}$, we can assume $c - ad = 1$. Then $x' = ax + y, y' = x + dx'$.

Therefore

$$
\begin{aligned}
&[\mathcal{O}_{\tilde{\mathcal{Z}}(x')} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y')}] \overset{\text{Case 1}}{=} [\mathcal{O}_{\tilde{\mathcal{Z}}(x')} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(x)}] = \left([\mathcal{O}_{\mathcal{Z}(x')}] - (n+1)[\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}]\right) \cdot [\mathcal{O}_{\mathcal{Z}(x)}] \\
&\overset{(52)}{=} [\mathcal{O}_{\mathcal{Z}(x')}] \cdot [\mathcal{O}_{\mathcal{Z}(x)}] - (n+1)\mathcal{O}(0,-1) = \Big([\mathcal{O}_{\tilde{\mathcal{Z}}(x)}] + (n+1)[\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}]\Big) \cdot [\mathcal{O}_{\mathcal{Z}(x')}] - \mathcal{O}(0,-n-1) \\
&\overset{(52)}{=} [\mathcal{O}_{\tilde{\mathcal{Z}}(x)}] \cdot [\mathcal{O}_{\mathcal{Z}(x')}] = [\mathcal{O}_{\tilde{\mathcal{Z}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(x')}] \overset{\text{Case 1}}{=} [\mathcal{O}_{\tilde{\mathcal{Z}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)}].
\end{aligned}
$$

□

Corollary 5.6.3. *Let $M \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 2. Let $\{x, y\}$ be a basis of M, then the elements $[\mathcal{O}_{\mathcal{Z}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)}]$ and $[\mathcal{O}_{\mathcal{Y}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Y}(y)}]$ belong to the group $\mathrm{Gr}^2 K_0^{\mathcal{Z}(M)}(\mathcal{M})$ and $\mathrm{Gr}^2 K_0^{\mathcal{Y}(M)}(\mathcal{M})$ respectively. Moreover, they only depend on the $\mathbb{Z}_p$-lattice M.*

Proof. Without loss of generality, we assume that $0 \leqslant \nu_p(q(x)) \leqslant \nu_p(q(y))$. For an arbitrary element $x' \in M$, we have $x' = ax + by$ for some elements $a, b \in \mathbb{Z}_p$. Then $q(x') = a^2 q(x) + ab(x,y) + b^2 q(y)$. Lemma 5.6.4 implies that $\nu_p((x,y)) \geqslant \nu_p(q(x))$, therefore $\nu_p(q(x')) \geqslant \nu_p(q(x))$. Hence x is a minimal element of M. Therefore the element $[\mathcal{O}_{\tilde{\mathcal{Z}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)}]$ only depends on M. Let $n = \nu_p(q(x)) = \min(M)$. By (52) and Lemma 5.1.1, we have

$$
\begin{aligned}
[\mathcal{O}_{\mathcal{Z}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)}] &= \Big([\mathcal{O}_{\tilde{\mathcal{Z}}(x)}] + (n+1)[\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}]\Big) \cdot [\mathcal{O}_{\mathcal{Z}(y)}] \\
&= [\mathcal{O}_{\tilde{\mathcal{Z}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)}] + \mathcal{O}(0,-n-1)
\end{aligned} \tag{55}
$$

Therefore the element $[\mathcal{O}_{\mathcal{Z}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)}]$ belongs to $\mathrm{Gr}^2 K_0^{\mathcal{Z}(M)}(\mathcal{M})$ and only depends on the $\mathbb{Z}_p$-lattice M by Proposition 5.6.2.

Notice that we have $\mathcal{Y}(x) = \left(\iota^{\mathcal{M}}\right)^* \left(\mathcal{Z}(x_0 \cdot x)\right)$ for all elements $x \in \mathbb{B}$. Hence $[\mathcal{O}_{\mathcal{Y}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Y}(y)}] = \left(\iota^{\mathcal{M}}\right)^* [\mathcal{O}_{\mathcal{Z}(x_0 \cdot x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(x_0 \cdot y)}]$ also depends on M only. □

Lemma 5.6.4. *Let $x, y \in \mathbb{B}$ be two elements. Then*

$$\nu_p((x,y)) \geqslant \min\{q(x), q(y)\}.$$

Proof. The statement actually works for all the anisotropic quadratic spaces. Suppose that $\nu_p(q(y)) \geqslant \nu_p(q(x))$. Let's assume the contrary that $\nu_p((x,y)) < \nu_p(q(x))$. Then the following equation over $\mathbb{Z}_p$

$$\frac{q(x)}{(x,y)} \cdot X^2 + X + \frac{q(y)}{(x,y)} = 0$$

must has a solution a in $p\mathbb{Z}_p$ by the Hensel's lemma. Then the vector $y' = y + ax$ is isotropic, which is a contradiction. Therefore $\nu_p((x,y)) \geqslant \nu_p(q(x)) = \min\{q(x), q(y)\}$. □

Let $L \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank r where $1 \leqslant r \leqslant 3$. Let $\boldsymbol{x} = \{x_1, \cdots, x_r\}$ be a basis of L. Define

$${}^{\mathbb{L}}\mathcal{Z}(\boldsymbol{x}) := [\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(x_r)}] \in \mathrm{Gr}^r K_0^{\mathcal{Z}(L)}(\mathcal{M}).$$

Notice that the element ${}^{\mathbb{L}}\mathcal{Z}(\boldsymbol{x})$ belongs to the r-graded piece $\mathrm{Gr}^r K_0^{\mathcal{Z}(L)}(\mathcal{M})$ because

- For $r = 1$, this follows from the definition of $\mathrm{Gr}^1 K_0^{\mathcal{Z}(L)}(\mathcal{M})$;
- For $r = 2$, this follows from the proof of Corollary 5.6.3;
- For $r = 3$, the element $[\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(x_2)}]$ is a linear combination of elements of the form $[\mathcal{O}_{W_s}]$, $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$ by Lemma 5.2.1, Lemma 5.6.1 and (55). The derived intersection

$[\mathcal{O}_{W_s} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{Z}(x_3)}]$ is proper by the theory of (quasi-)canonical liftings [Gro86], hence it belongs to the graded piece $\mathrm{Gr}^3 K_0^{\mathcal{Z}(L)}(\mathcal{M})$. The derived intersection $\mathcal{O}(1,0)\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}}[\mathcal{O}_{\mathcal{Z}(x_3)}] = \mathcal{O}(1,0)\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathrm{Exc}_\mathcal{M}}}[\mathcal{O}_{\mathrm{Exc}_\mathcal{M}}]\otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}}[\mathcal{O}_{\mathcal{Z}(x_3)}] = \mathcal{O}(1,0)\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathrm{Exc}_\mathcal{M}}}\mathcal{O}(0,-1)$ by Corollary 5.3.2(b), hence it belongs to the graded piece $\mathrm{Gr}^3 K_0^{\mathcal{Z}(L)}(\mathcal{M})$. Similar argument also applies to the derived intersection $\mathcal{O}(0,1) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} [\mathcal{O}_{\mathcal{Z}(x_3)}]$.

We also define

$$^{\mathbb{L}}\mathcal{Y}(\boldsymbol{x}) := [\mathcal{O}_{\mathcal{Y}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{Y}(x_r)}] \in \mathrm{Gr}^r K_0^{\mathcal{Y}(L)}(\mathcal{M}).$$

Notice that the elements $^{\mathbb{L}}\mathcal{Z}(\boldsymbol{x})$ and $^{\mathbb{L}}\mathcal{Y}(\boldsymbol{x})$ are invariant under permutations and operations of the form $x_i \to a_i x_i + a_j x_j$ for some $a_i \in \mathbb{Z}_p^\times$ and $a_j \in \mathbb{Z}_p$ by Corollary 5.6.3. We can also transform the basis $\boldsymbol{x}$ by permutations and operations of the form $x_i \to a_i x_i + a_j x_j$ for some $a_i \in \mathbb{Z}_p^\times$ and $a_j \in \mathbb{Z}_p$ to get any another basis $\boldsymbol{x}' = \{x_1', \cdots, x_r'\}$ of L. Therefore the elements $^{\mathbb{L}}\mathcal{Z}(\boldsymbol{x})$ and $^{\mathbb{L}}\mathcal{Y}(\boldsymbol{x})$ only depend on L.

Definition 5.6.5. Let $L \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank r where $1 \leqslant r \leqslant 3$. Let $\{x_1, \cdots, x_r\}$ be a basis of L. Define the derived special cycle

$$^{\mathbb{L}}\mathcal{Z}(L) := [\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{Z}(x_r)}] \in \mathrm{Gr}^r K_0^{\mathcal{Z}(L)}(\mathcal{M}).$$
$$^{\mathbb{L}}\mathcal{Y}(L) := [\mathcal{O}_{\mathcal{Y}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{Y}(x_r)}] \in \mathrm{Gr}^r K_0^{\mathcal{Y}(L)}(\mathcal{M}).$$

Definition 5.6.6. Let $L \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 3. Define the arithmetic intersection numbers

$$\mathrm{Int}^{\mathcal{Z}}(L) := \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L)), \quad \mathrm{Int}^{\mathcal{Y}}(L) := \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Y}(L))$$

Here χ denotes the Euler–Poincaré characteristic.

Now we are able to state to state the main theorem of the article.

Theorem 5.6.7. *Let $L \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 3. Then*

$$\mathrm{Int}^{\mathcal{Z}}(L) = \partial\mathrm{Den}\left(H_0(p), L\right),$$

and

$$\mathrm{Int}^{\mathcal{Y}}(L) = \partial\mathrm{Den}\left(H_0(p)^\vee, L\right) - 1 = \partial\mathrm{Den}\left(H_0(p)^\vee, L\right) - \frac{p^7}{2(p+1)^2} \cdot \mathrm{Den}(\mathcal{O}_{\mathbb{B}}^\vee, L).$$

6. Difference formula on the geometric side

6.1. **Difference formula: the hyperspecial case.** Let $L \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank r where $1 \leqslant r \leqslant 3$. Let $\{x_1, \cdots, x_r\}$ be a basis of L. Then the element $\mathcal{O}_{\mathcal{Z}_\mathcal{N}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}_\mathcal{N}(x_r)} \in \mathrm{F}^r K_0^{\mathcal{Z}_\mathcal{N}(L)}(\mathcal{N})$ because it is a proper intersection. The element $[\mathcal{O}_{\mathcal{Z}_\mathcal{N}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}_\mathcal{N}(x_r)}] \in \mathrm{Gr}^r K_0^{\mathcal{Z}_\mathcal{N}(L)}(\mathcal{N})$ only depends on the lattice L [LZ22b, Corollary 4.11.2]. Denote by $^{\mathbb{L}}\mathcal{Z}_\mathcal{N}(L)$ the image of this element in $\mathrm{Gr}^r K_0^{\mathcal{Z}(L)}(\mathcal{N})$. If the rank of L is 3, then we define the arithmetic intersection number on $\mathcal{N}$ as

$$\mathrm{Int}_\mathcal{N}(L) := \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}_\mathcal{N}(L)).$$

The works of Gross–Keating [GK93] and the ARGOS volume [VGW+07] together imply the following theorem for all the prime number p (including $p = 2$):

Theorem 6.1.1. *Let $L \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 3. Then*

$$\mathrm{Int}_{\mathcal{N}}(L) = \partial\mathrm{Den}(H, L).$$

Here

$$\partial\mathrm{Den}(H, L) := -\frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \frac{\mathrm{Den}(X, H_4^+, L)}{\mathrm{Den}(H_4^+, H_3^+)},$$

where H_{2k}^+ is as introduced in the introduction and $H_3^+ = H_2^+ \oplus H_1^+$.

Lemma 6.1.2. *Let $\mathcal{N}_{\mathbb{F}} := \mathcal{N} \times_W \mathbb{F}$. Let $L^\flat \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 2. Let $x \in \mathbb{B}$ be another element such that $\nu_p(q(x)) \geqslant \max\{\max(L^\flat), 2\}$ and $x \perp L^\flat$, then*

$$\begin{aligned}\mathrm{Int}_{\mathcal{N}}(L^\flat \oplus \langle x\rangle) - \mathrm{Int}_{\mathcal{N}}(L^\flat \oplus \langle p^{-1}x\rangle) &= \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}_{\mathcal{N}}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{N}_{\mathbb{F}}})\\ &= \partial\mathrm{Den}\left(\langle x\rangle[-1] \oplus H_2^+, L^\flat\right).\end{aligned}$$

Proof. The first identity is proved by [GK93, Lemma 5.6] and [Rap05, Proposition 1.6]. For the second identity, we combine Theorem 6.1.1 and the following identity by [Zhu25, Theorem 7.2.6]:

$$\partial\mathrm{Den}(H, L^\flat \oplus \langle x\rangle) - \partial\mathrm{Den}(H, L^\flat \oplus \langle p^{-1}x\rangle) = \partial\mathrm{Den}\left(\langle x\rangle[-1] \oplus H_2^+, L^\flat\right).$$

□

6.2. **Blow up: the hyperspecial case.** Recall that $\mathcal{N} \simeq \mathrm{Spf}\, W[[t, t']]$, here the elements t and t' are chosen so that under the identification $\mathcal{N} = \mathcal{N}_0 \times_W \mathcal{N}_0$, the first $\mathcal{N}_0 \simeq \mathrm{Spf}\, W[[t]]$ and the second $\mathcal{N}_0 \simeq \mathrm{Spf}\, W[[t']]$. Hence $\mathcal{N}_{\mathbb{F}} \simeq \mathrm{Spf}\, \mathbb{F}[[t, t']]$. Let $\pi_{\mathbb{F}} : \widetilde{\mathcal{N}}_{\mathbb{F}} \to \mathcal{N}_{\mathbb{F}}$ be the blow up morphism along the unique closed point of $\mathcal{N}_{\mathbb{F}}$. There is an open cover $\{\widetilde{\mathcal{N}}_{\mathbb{F}}^\circ, \widetilde{\mathcal{N}}_{\mathbb{F}}^\bullet\}$ of $\widetilde{\mathcal{N}}_{\mathbb{F}}$ given as follows,

- ∘ Let $t' = xt$. Define $\widetilde{\mathcal{N}}_{\mathbb{F}}^\circ = \mathrm{Spf}\, \mathbb{F}[x][[t]]$.
- • Let $t = x't'$. Define $\widetilde{\mathcal{N}}_{\mathbb{F}}^\bullet = \mathrm{Spf}\, \mathbb{F}[x'][[t']]$.

Over the intersection $\widetilde{\mathcal{N}}_{\mathbb{F}}^\circ \times_{\mathcal{N}} \widetilde{\mathcal{N}}_{\mathbb{F}}^\bullet$, we have $x'x = 1$. Let $\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}$ be the exceptional divisor on $\widetilde{\mathcal{N}}_{\mathbb{F}}$. It is glued by $\mathrm{Spec}\, \mathbb{F}[x]$ and $\mathrm{Spec}\, \mathbb{F}[x']$ with the condition $x'x = 1$. Hence $\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}} \simeq \mathbb{P}^1_{\mathbb{F}}$.

Lemma 6.2.1. *We have the following identity in $\mathrm{Gr}^2 K_0^{\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}}(\widetilde{\mathcal{N}}_{\mathbb{F}}) \simeq \mathrm{Pic}(\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}})$:*

$$[\mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}} \mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}}] = \mathcal{O}(-1). \tag{56}$$

Proof. Similar to Lemma 5.2.1, the derived tensor product $[\mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}} \mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}}]$ can be viewed as the restriction of the line bundle corresponding to the exceptional divisor $\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}$ on $\widetilde{\mathcal{N}}_{\mathbb{F}}$ to the exceptional divisor $\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}$ itself. Under the open cover $\widetilde{\mathcal{N}}_{\mathbb{F}}^\circ$ and $\widetilde{\mathcal{N}}_{\mathbb{F}}^\bullet$ of $\widetilde{\mathcal{N}}_{\mathbb{F}}$, the divisor $\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}$ is given by the following equations and transformation rules:

$$\begin{array}{ccc} \widetilde{\mathcal{N}}_{\mathbb{F}}^\circ & & \widetilde{\mathcal{N}}_{\mathbb{F}}^\bullet \\ \Cup & & \Cup \\ t & \underset{\times x'}{\overset{\times x}{\rightleftarrows}} & t' \end{array}$$

The same transformation rule also applies to the corresponding open cover of $\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}} \simeq \mathbb{P}^1_{\mathbb{F}}$. Therefore the corresponding line bundle is $\mathcal{O}(-1)$. □

6.3. **Invariance of the intersection number.** Let $x \in \mathbb{B}$ be a nonzero element. Define $\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x) := \mathcal{Z}_{\mathcal{N}}(x) \times_{\mathcal{N}} \mathcal{N}_{\mathbb{F}}$. The intersection is proper by the flatness over W of the divisor $\mathcal{Z}_{\mathcal{N}}(x)$ (Lemma 4.3.2), hence $\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x)$ is an effective Cartier divisor on $\mathcal{N}_{\mathbb{F}}$. Define $\mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(x) = \pi_{\mathbb{F}}^{*}\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x) = \widetilde{\mathcal{N}}_{\mathbb{F}} \times_{\mathcal{N}_{\mathbb{F}}} \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x)$ to be the pull back of the divisor $\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x)$ on $\mathcal{N}_{\mathbb{F}}$ to the blow up $\widetilde{\mathcal{N}}_{\mathbb{F}}$.

Lemma 6.3.1. *Let $x \in \mathbb{B}$ be a nonzero element such that $n := \nu_p(q(x)) \geqslant 0$. Then*

(a) *There exists an isomorphism $\mathcal{N}_{\mathbb{F}} \simeq \operatorname{Spf} \mathbb{F}[[t,t']]$ such that every irreducible component of $\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x)$ is of the form*

$$\mathcal{C}_a(x) \simeq \operatorname{Spf} \mathbb{F}[[t,t']]/(\nu_x' t' - t^{p^a}) \text{ or } \mathcal{C}_{-a}(x) \simeq \operatorname{Spf} \mathbb{F}[[t,t']]/(\nu_x t - t'^{p^a}),$$

where $0 \leqslant a \leqslant n$ and $a \equiv n \bmod 2$, $\nu_x, \nu_x' \in \mathcal{O}_{\mathcal{N}_{\mathbb{F}}}^{\times}$, here $\mathcal{C}_a = \mathcal{C}_{a'}$ if and only if $a = a'$. Moreover, we have the following identity of effective Cartier divisors on $\mathcal{N}_{\mathbb{F}}$,

$$\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x) = \sum_{\substack{-n \leqslant a \leqslant n \\ a \equiv n \bmod 2}} p^{(n-|a|)/2} \cdot \mathcal{C}_a(x).$$

(b) *We have $\operatorname{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}} \simeq \mathbb{P}^1_{\mathbb{F}}$ and the following equality in the group $\operatorname{Pic}(\operatorname{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}})$:*

$$[\mathcal{O}_{\mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}} \mathcal{O}_{\operatorname{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}}] = \mathcal{O}(0). \tag{57}$$

(c) *Let $y \in \mathbb{B}$ be another element such that x, y are linearly independent. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be an irreducible component of $\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x)$ and $\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(y)$ respectively. Then*

$$\chi(\widetilde{\mathcal{N}}_{\mathbb{F}}, \mathcal{O}_{\pi_{\mathbb{F}}^{*}\mathcal{C}_1} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}} \mathcal{O}_{\pi_{\mathbb{F}}^{*}\mathcal{C}_2}) = \chi(\mathcal{N}_{\mathbb{F}}, \mathcal{O}_{\mathcal{C}_1} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_{\mathbb{F}}}} \mathcal{O}_{\mathcal{C}_2}). \tag{58}$$

Proof. Part (a) is a consequence of Proposition 4.4.1 (b2). Let $\mathcal{C}$ be an irreducible component of a special divisor $\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x)$, we have the following equality by (a):

$$\pi_{\mathbb{F}}^{*}\mathcal{C} = \operatorname{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}} + \widetilde{\mathcal{C}},$$

where $\widetilde{\mathcal{C}}$ is the strict transform of the divisor $\mathcal{C}$ under the blow up morphism $\pi_{\mathbb{F}}$. It is isomorphic to $\mathcal{C}$ under the morphism $\pi_{\mathbb{F}}$ since it is represented by a 1-dimensional regular local ring. It's easy to see that $\widetilde{\mathcal{C}} \cap \operatorname{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}} := \widetilde{\mathcal{C}} \times_{\widetilde{\mathcal{N}}_{\mathbb{F}}} \operatorname{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}$ is scheme-theoretically a single point scheme, hence we have the equality $[\mathcal{O}_{\widetilde{\mathcal{C}} \cap \operatorname{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}}] = \mathcal{O}(1)$, hence $[\mathcal{O}_{\pi_{\mathbb{F}}^{*}\mathcal{C} \cap \operatorname{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}}] = \mathcal{O}(0)$, where $\pi_{\mathbb{F}}^{*}\mathcal{C} \cap \operatorname{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}} := \pi_{\mathbb{F}}^{*}\mathcal{C} \times_{\widetilde{\mathcal{N}}_{\mathbb{F}}} \operatorname{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}$. Therefore (b) is true since the special divisor $\mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(x)$ is a summation of divisors of the form in (a). Moreover,

$$\begin{aligned}
\chi(\widetilde{\mathcal{N}}_{\mathbb{F}}, \mathcal{O}_{\pi_{\mathbb{F}}^{*}\mathcal{C}_1} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}} \mathcal{O}_{\pi_{\mathbb{F}}^{*}\mathcal{C}_2}) &= (\pi_{\mathbb{F}}^{*}\mathcal{C}_1 \cdot \pi_{\mathbb{F}}^{*}\mathcal{C}_2)_{\widetilde{\mathcal{N}}_{\mathbb{F}}} = \Big((\widetilde{\mathcal{C}}_1 + \operatorname{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}) \cdot \pi_{\mathbb{F}}^{*}\mathcal{C}_2\Big)_{\widetilde{\mathcal{N}}_{\mathbb{F}}} \\
&= \Big(\widetilde{\mathcal{C}}_1 \cdot \pi_{\mathbb{F}}^{*}\mathcal{C}_2\Big)_{\widetilde{\mathcal{N}}_{\mathbb{F}}} \overset{\text{projection formula}}{=} (\mathcal{C}_1 \cdot \mathcal{C}_2)_{\mathcal{N}_{\mathbb{F}}} \\
&= \chi(\mathcal{N}_{\mathbb{F}}, \mathcal{O}_{\mathcal{C}_1} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_{\mathbb{F}}}} \mathcal{O}_{\mathcal{C}_2}).
\end{aligned}$$

□

Corollary 6.3.2. *Let $x, y \in \mathbb{B}$ be two linearly independent elements. Then*

$$\chi(\widetilde{\mathcal{N}}_{\mathbb{F}}, \mathcal{O}_{\mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}} \mathcal{O}_{\mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(y)}) = \chi(\mathcal{N}_{\mathbb{F}}, \mathcal{O}_{\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_{\mathbb{F}}}} \mathcal{O}_{\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(y)}). \tag{59}$$

Proof. Since $\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x)$ is a summation of regular (smooth) divisors $\mathcal{C}$. The equality follows from Lemma 6.3.1 (b). □

6.4. Geometric difference formula on $\mathcal{M}$.

Lemma 6.4.1. *Let $L^\flat \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 2. Let $x \in \mathbb{B}$ be another element such that $\nu_p(q(x)) \geqslant \max\{\mathrm{val}(L^\flat), 2\}$ and $x \perp L^\flat$, then*

$$\begin{aligned}\mathrm{Int}^{\mathcal{Z}}(L^\flat \oplus \langle x\rangle) - \mathrm{Int}^{\mathcal{Z}}(L^\flat \oplus \langle p^{-1}x\rangle) &= \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}_{\mathbb{F}}}) \\ &= \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}^{\mathrm{FF}}}) + \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}^{\mathrm{VV}}}) \\ &\quad + \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}^{\mathrm{FV}}}) + \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}^{\mathrm{VF}}}).\end{aligned}$$

Proof. By Lemma 5.1.1, we know that

$$\mathrm{Int}^{\mathcal{Z}}(L^\flat \oplus \langle x\rangle) - \mathrm{Int}^{\mathcal{Z}}(L^\flat \oplus \langle p^{-1}x\rangle) = \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{D}(x)}).$$

Therefore the first equality is equivalent to

$$\chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{D}(x)}) = \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}_{\mathbb{F}}}). \tag{60}$$

By Lemma 5.6.1 and the isomorphism $\mathrm{Pic}(\mathrm{Exc}_{\mathcal{M}}) \simeq \mathrm{Gr}^2 K_0^{\mathrm{Exc}_{\mathcal{M}}}(\mathcal{M})$ given by $\mathcal{L} \mapsto [\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}] - [\mathcal{L}]$,

$${}^{\mathbb{L}}\mathcal{Z}(L^\flat) = \text{linear combinations of } \mathcal{O}(1,0),\ \mathcal{O}(0,1) \text{ and } [\mathcal{O}_{W_s}].$$

For elements of the form $[\mathcal{O}_{W_s}]$, by the moduli interpretation of the special divisor $\mathcal{Z}(x)$ and [GK93, Lemma 5.11], we have

$$\chi(\mathcal{M}, [\mathcal{O}_{W_s}] \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{D}(x)}) = \chi(\mathcal{M}, [\mathcal{O}_{W_s}] \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}_{\mathbb{F}}}).$$

For the elements $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$, we have

$$\chi(\mathcal{M}, \mathcal{O}(1,0) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{D}(x)}) = \chi(\mathcal{M}, \mathcal{O}(0,1) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{D}(x)}) = 0$$

by Proposition 5.3.1. On the other hand, by Proposition 4.11.1 (iii), we have

$$\chi(\mathcal{M}, \mathcal{O}(1,0) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}_{\mathbb{F}}}) = \chi(\mathcal{M}, \mathcal{O}(0,1) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}_{\mathbb{F}}}) = 0.$$

Therefore (60) is true and hence the first equality in the Lemma.

For the second equality: Let x_1, x_2 be a basis of L, we have

$$\begin{aligned}{}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}} &= [\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(x_2)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}] \\ &= \Big([\mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}]\Big) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}} \Big([\mathcal{O}_{\mathcal{Z}(x_2)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}]\Big) \\ &= \mathcal{O}(0,-1) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}(0,-1) = 0.\end{aligned}$$

Therefore

$$\begin{aligned}{}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}_{\mathbb{F}}} &= {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} (2\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}} + \mathcal{O}_{\mathcal{M}^{\mathrm{FF}}} + \mathcal{O}_{\mathcal{M}^{\mathrm{VV}}} + \mathcal{O}_{\mathcal{M}^{\mathrm{FV}}} + \mathcal{O}_{\mathcal{M}^{\mathrm{FV}}}) \\ &= {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} (\mathcal{O}_{\mathcal{M}^{\mathrm{FF}}} + \mathcal{O}_{\mathcal{M}^{\mathrm{VV}}} + \mathcal{O}_{\mathcal{M}^{\mathrm{FV}}} + \mathcal{O}_{\mathcal{M}^{\mathrm{FV}}}).\end{aligned}$$

Therefore the second equality is true. □

6.5. **Intersections on $\mathcal{M}^{\mathbf{FF}}$.** We defined $p_+ : \mathcal{M}^+ \to \mathcal{N}$ as a composition $\mathcal{M}^+ \to \mathcal{M} \xrightarrow{\pi} \mathcal{N}(x_0) \xrightarrow{s_+} \mathcal{N}$. Notice that $\mathcal{M}^{\mathrm{FF}}$ is a closed formal subscheme of $\mathcal{M}^+_{\mathbb{F}}$. Let $\pi^{\mathrm{FF}} : \mathcal{M}^{\mathrm{FF}} \to \mathcal{N}_{\mathbb{F}}$ be the composition $\mathcal{M}^{\mathrm{FF}} \hookrightarrow \mathcal{M}^+_{\mathbb{F}} \xrightarrow{(p_+)_{\mathbb{F}}} \mathcal{N}_{\mathbb{F}}$. Using the open cover we fixed in §4.11 and Lemma 4.6.4, we have the following isomorphism of formal schemes over $\mathcal{N}_{\mathbb{F}}$:

$$\iota^{\mathrm{FF}} : \mathcal{M}^{\mathrm{FF}} \xrightarrow{\sim} \widetilde{\mathcal{N}}_{\mathbb{F}}. \tag{61}$$

Let $x \in \mathbb{B}$ be a nonzero element. By the moduli interpretations of the divisor $\mathcal{Z}^+(x)$ on $\mathcal{M}^+$ in Lemma 4.13.2 and the isomorphism $\iota^{\mathrm{FF}} : \mathcal{M}^{\mathrm{FF}} \to \widetilde{\mathcal{N}}_{\mathbb{F}}$, we have

$$\mathcal{Z}(x) \cap \mathcal{M}^{\mathrm{FF}} = \left(\iota^{\mathrm{FF}}\right)^* \mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(x) := \mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(x) \times_{\widetilde{\mathcal{N}}_{\mathbb{F}},\iota^{\mathrm{FF}}} \mathcal{M}^{\mathrm{FF}} = \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x) \times_{\mathcal{N}_{\mathbb{F}},\pi^{\mathrm{FF}}} \mathcal{M}^{\mathrm{FF}}. \tag{62}$$

Lemma 6.5.1. *Let $L^\flat \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$\chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}^{\mathrm{FF}}}) = \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}_{\mathcal{N}}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{N}_{\mathbb{F}}}).$$

Proof. Let x, y be a $\mathbb{Z}_p$-basis of the lattice $L^\flat$. Then

$$\begin{aligned}\chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}^{\mathrm{FF}}}) &= \chi(\mathcal{M}^{\mathrm{FF}}, \mathcal{O}_{\mathcal{Z}(x)\cap\mathcal{M}^{\mathrm{FF}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}^{\mathrm{FF}}}} \mathcal{O}_{\mathcal{Z}(y)\cap\mathcal{M}^{\mathrm{FF}}}) \\ &\overset{(62)}{=} \chi(\widetilde{\mathcal{N}}_{\mathbb{F}}, \mathcal{O}_{\mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}} \mathcal{O}_{\mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(y)}) \\ &\overset{(59)}{=} \chi(\mathcal{N}_{\mathbb{F}}, \mathcal{O}_{\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_{\mathbb{F}}}} \mathcal{O}_{\mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(y)}) \\ &= \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}_{\mathcal{N}}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{N}_{\mathbb{F}}}).\end{aligned}$$

□

6.6. **Intersections on $\mathcal{M}^{\mathbf{VV}}$.** The ideas of the computations in this part is similar to §6.5. We have defined $p_- : \mathcal{M}^- \to \mathcal{N}$ as a composition $\mathcal{M}^- \to \mathcal{M} \xrightarrow{\pi} \mathcal{N}(x_0) \xrightarrow{s_-} \mathcal{N}$. Notice that $\mathcal{M}^{\mathrm{VV}}$ is a closed formal subscheme of $\mathcal{M}^-_{\mathbb{F}}$. Let $\pi^{\mathrm{VV}} : \mathcal{M}^{\mathrm{VV}} \to \mathcal{N}_{\mathbb{F}}$ be the composition $\mathcal{M}^{\mathrm{VV}} \hookrightarrow \mathcal{M}^-_{\mathbb{F}} \xrightarrow{(p_-)_{\mathbb{F}}} \mathcal{N}_{\mathbb{F}}$. Using the open cover we fixed in §4.11 and Lemma 4.6.4, we have the following isomorphism of formal schemes over $\mathcal{N}_{\mathbb{F}}$:

$$\iota^{\mathrm{VV}} : \mathcal{M}^{\mathrm{VV}} \xrightarrow{\sim} \widetilde{\mathcal{N}}_{\mathbb{F}}. \tag{63}$$

Let $x \in \mathbb{B}$ be a nonzero element. By the moduli interpretations of the divisor $\mathcal{Z}^-(x)$ on $\mathcal{M}^-$ in Lemma 4.13.2 and the isomorphism $\iota^{\mathrm{VV}} : \mathcal{M}^{\mathrm{VV}} \to \widetilde{\mathcal{N}}_{\mathbb{F}}$, we have

$$\mathcal{Z}(x) \cap \mathcal{M}^{\mathrm{VV}} = \left(\iota^{\mathrm{VV}}\right)^* \mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(x') := \mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(x') \times_{\widetilde{\mathcal{N}}_{\mathbb{F}},\iota^{\mathrm{VV}}} \mathcal{M}^{\mathrm{FF}} = \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x') \times_{\mathcal{N}_{\mathbb{F}},\pi^{\mathrm{VV}}} \mathcal{M}^{\mathrm{VV}}. \tag{64}$$

Lemma 6.6.1. *Let $L^\flat \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 2. Let $L^{\flat\prime} \subset \mathbb{B}$ be the image of $L^\flat$ under the isometric homomorphism $(\cdot)' : \mathbb{B} \to \mathbb{B}$. Then*

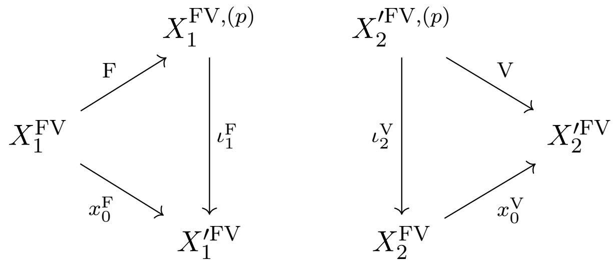
$$\chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}^{\mathrm{VV}}}) = \chi(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}_{\mathcal{N}}(L^{\flat\prime}) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{N}_{\mathbb{F}}}).$$

Proof. The proof is the same as that of Lemma 6.5.1, we just need to replace $\mathcal{M}^{\mathrm{FF}}$ there by $\mathcal{M}^{\mathrm{VV}}$. □

6.7. **Intersections on $\mathcal{M}^{\mathbf{FV}}$.** Let

$$\left(X_1^{\mathrm{FV}} \xrightarrow{x_0^{\mathrm{F}}} X_1'^{\mathrm{FV}}, \left(\rho_1^{\mathrm{FV}}, \rho_1'^{\mathrm{FV}}\right)\right), \quad \left(X_2^{\mathrm{FV}} \xrightarrow{x_0^{\mathrm{V}}} X_2'^{\mathrm{FV}}, \left(\rho_2^{\mathrm{FV}}, \rho_2'^{\mathrm{FV}}\right)\right) \tag{65}$$

be the base change of the universal object (38) over $\mathcal{M}$ to $\mathcal{M}^{\mathrm{FV}}$. By the equations of $\mathcal{M}^{\mathrm{FV}}$ in Proposition 4.11.1, the morphism x_0^{F} is isomorphic to the Frobenius morphism, while x_0^{V} is isomorphic to the Verschiebung morphism. Hence there exist two isomorphisms $\iota_1^{\mathrm{F}} : X_1^{\mathrm{FV},(p)} \to X_1'^{\mathrm{FV}}$ and $\iota_2^{\mathrm{V}} : X_2'^{\mathrm{FV},(p)} \to X_2^{\mathrm{FV}}$ over $\mathcal{M}^{\mathrm{FV}}$ such that the following diagrams commute,



here we use F, V to represent the relative Frobenius and Verschiebung morphism.

For a morphism $f : S \to \mathcal{M}^{\mathrm{FV}}$. Let $(X_{1,S}^{\mathrm{FV}}, \rho_{1,S}^{\mathrm{FV}})$ and $(X_{2,S}'^{\mathrm{FV}}, \rho_{2,S}'^{\mathrm{FV}})$ be the base change of the p-divisible groups $(X_1^{\mathrm{FV}}, \rho_1^{\mathrm{FV}})$ and $(X_2'^{\mathrm{FV}}, \rho_{2,S}'^{\mathrm{FV}})$ to S through the morphism f. Then we get a map $\mathcal{M}^{\mathrm{FV}}(S) \to \mathcal{N}_{\mathbb{F}}(S) : f \mapsto \left(\left(X_{1,S}^{\mathrm{FV}}, \rho_{1,S}^{\mathrm{FV}}\right), \left(X_2'^{\mathrm{FV}}, \rho_{2,S}'^{\mathrm{FV}}\right)\right)$. Therefore we obtain a morphism between formal schemes $\pi^{\mathrm{FV}} : \mathcal{M}^{\mathrm{FV}} \to \mathcal{N}_{\mathbb{F}}$. Using the open cover we fixed in §4.11 and Lemma 4.6.4, we have the following isomorphism of formal schemes over $\mathcal{N}_{\mathbb{F}}$:

$$\iota^{\mathrm{FV}} : \mathcal{M}^{\mathrm{FV}} \xrightarrow{\sim} \widetilde{\mathcal{N}}_{\mathbb{F}}. \tag{66}$$

Denote by $\mathcal{M}_{\circ}^{\mathrm{FV}} = \left(\iota^{\mathrm{FV}}\right)^* \left(\widetilde{\mathcal{N}}_{\mathbb{F}}^{\circ}\right)$, $\mathcal{M}_{\bullet}^{\mathrm{VF}} = \left(\iota^{\mathrm{FV}}\right)^* \left(\widetilde{\mathcal{N}}_{\mathbb{F}}^{\bullet}\right)$. Then $\mathcal{M}_{\circ}^{\mathrm{FV}} \subset \mathcal{M}^+$, $\mathcal{M}_{\bullet}^{\mathrm{FV}} \subset \mathcal{M}^-$ and $\{\mathcal{M}_{\circ}^{\mathrm{FV}}, \mathcal{M}_{\bullet}^{\mathrm{FV}}\}$ gives an open cover of $\mathcal{M}^{\mathrm{FV}}$.

Let $\mathcal{N}_{0,\mathbb{F}} = \mathcal{N}_0 \times_W \mathbb{F}$. Denote by $\mathrm{Fr} : \mathcal{N}_{0,\mathbb{F}} \to \mathcal{N}_{0,\mathbb{F}}$ the relative Frobenius morphism of $\mathcal{N}_{0,\mathbb{F}}$. The morphism sends a pair $(X, \rho) \in \mathcal{N}_{0,\mathbb{F}}(S)$ to the pair $(X^{(p)}, \rho^{(p)})$, here $X^{(p)} = X \times_{S,\mathrm{Fr}_S} S$ where $\mathrm{Fr}_S : S \to S$ is the absolute Frobenius morphism. Denote by $p_{\circ}^{\mathrm{FV}} : \mathcal{M}_{\circ}^{\mathrm{FV}} \to \mathcal{N}_{\mathbb{F}}$ the restriction of the morphism $p_+ : \mathcal{M}^+ \to \mathcal{N}$ (cf. (39)) to $\mathcal{M}_{\circ}^{\mathrm{FV}}$. Denote by $p_{\bullet}^{\mathrm{FV}} : \mathcal{M}_{\bullet}^{\mathrm{FV}} \to \mathcal{N}_{\mathbb{F}}$ the restriction of the morphism $p_- : \mathcal{M}^- \to \mathcal{N}$ (cf. (39)) to $\mathcal{M}_{\bullet}^{\mathrm{FV}}$. Then we have

$$p_{\circ}^{\mathrm{FV}} = (\mathrm{Id} \times \mathrm{Fr}) \circ \pi^{\mathrm{FV}}, \quad p_{\bullet}^{\mathrm{FV}} = (\mathrm{Fr} \times \mathrm{Id}) \circ \pi^{\mathrm{FV}}. \tag{67}$$

Let $x \in \mathbb{B}$ be a nonzero element such that $\nu_p(q(x)) \geqslant 0$. By the moduli interpretation of the divisor $\mathcal{Z}(x)$ in Lemma 4.13.2, we have

$$\mathcal{Z}(x) \cap \mathcal{M}_{\circ}^{\mathrm{FV}} = \left(\pi^{\mathrm{FV}}\right)^* (\mathrm{Id} \times \mathrm{Fr})^* \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x), \ \mathcal{Z}(x) \cap \mathcal{M}_{\bullet}^{\mathrm{FV}} = \left(\pi^{\mathrm{FV}}\right)^* (\mathrm{Fr} \times \mathrm{Id})^* \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x'), \tag{68}$$

Notice that the quasi-isogeny $x_0 \cdot x = x' \cdot x_0 : X_1^{\mathrm{FV}} \dashrightarrow X_2'^{\mathrm{FV}}$ lifts to an isogeny over $\mathcal{Z}(x) \cap \mathcal{M}^{\mathrm{FV}}$. Therefore

$$\mathcal{Z}(x) \cap \mathcal{M}_{\circ}^{\mathrm{FV}} \subset \left(\pi^{\mathrm{FV}}\right)^* \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x_0 \cdot x), \ \mathcal{Z}(x) \cap \mathcal{M}_{\bullet}^{\mathrm{FV}} \subset \left(\pi^{\mathrm{FV}}\right)^* \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x' \cdot x_0). \tag{69}$$

Lemma 6.7.1. *Let $x \in \mathbb{B}$ be a nonzero element such that $\nu_p(q(x)) \geqslant 0$. Then we have the following identity of effective Cartier divisors on $\mathcal{M}^{\mathrm{FV}}$:*

$$\mathcal{Z}(x) \cap \mathcal{M}^{\mathrm{FV}} = \left(\iota^{\mathrm{FV}}\right)^* \left(p \cdot \mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(\overline{x_0}^{-1} \cdot x) + \mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}\right). \tag{70}$$

Proof. Let $n = \nu_p(q(x)) \geqslant 0$. Since x_0^{F} (resp. x_0^{V}) is isomorphic to the relative Frobenius (resp. Verschiebung) morphism, we have the following equality:

$$(\mathrm{Id} \times \mathrm{Fr})^* \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x) = \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x_0 \cdot x) - \mathcal{C}_{n+1}(x_0 \cdot x) = p \cdot \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(\overline{x_0}^{-1} \cdot x) + \mathcal{C}_{-n-1}(x_0 \cdot x),$$
$$(\mathrm{Fr} \times \mathrm{Id})^* \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x) = \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x' \cdot x_0) - \mathcal{C}_{-n-1}(x' \cdot x_0) = p \cdot \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(\overline{x_0}^{-1} \cdot x) + \mathcal{C}_{n+1}(x' \cdot x_0).$$

Here we use the equality $p^{-1}x_0 \cdot x = \overline{x_0}^{-1} \cdot x$. By (68) and (69) and the equation of $\mathcal{C}_{-n-1}(x_0 \cdot x)$ in Lemma 6.3.1(a), we have that over the open formal subscheme $\mathcal{M}_{\circ}^{\mathrm{FV}}$,

$$\begin{aligned}\mathcal{Z}(x) \cap \mathcal{M}_{\circ}^{\mathrm{FV}} &= \left(\pi^{\mathrm{FV}}\right)^* \left(p \cdot \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(\overline{x_0}^{-1} \cdot x) + \mathcal{C}_{-n-1}(x_0 \cdot x)\right) \\ &= \left(\iota^{\mathrm{FV}}\right)^* \left(p \cdot \mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(\overline{x_0}^{-1} \cdot x) + \mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}\right).\end{aligned}$$

While over the open formal subscheme $\mathcal{M}_{\bullet}^{\mathrm{FV}}$,

$$\begin{aligned}\mathcal{Z}(x) \cap \mathcal{M}_{\bullet}^{\mathrm{FV}} &= \left(\pi^{\mathrm{FV}}\right)^* \left(p \cdot \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(\overline{x_0}^{-1} \cdot x) + \mathcal{C}_{n+1}(x' \cdot x_0)\right) \\ &= \left(\iota^{\mathrm{FV}}\right)^* \left(p \cdot \mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(\overline{x_0}^{-1} \cdot x) + \mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}\right).\end{aligned}$$

The formula (70) is true on an open cover of $\mathcal{M}^{\mathrm{FV}}$, hence is true over all $\mathcal{M}^{\mathrm{FV}}$. $\square$

Corollary 6.7.2. *Let $L^\flat \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$\chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}^{\mathrm{FV}}}) = p^2 \cdot \chi\left(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}_{\mathcal{N}}\left(\overline{x_0}^{-1} \cdot L^\flat\right) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{N}_{\mathbb{F}}}\right) - 1.$$

Proof. Let x, y be a $\mathbb{Z}_p$-basis of the lattice $L^\flat$. We have

$$\begin{aligned}\chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}^{\mathrm{FV}}}) &= \chi(\mathcal{M}^{\mathrm{FV}}, \mathcal{O}_{\mathcal{Z}(x)\cap\mathcal{M}^{\mathrm{FV}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}^{\mathrm{FV}}}} \mathcal{O}_{\mathcal{Z}(x)\cap\mathcal{M}^{\mathrm{FV}}}) \\ &\overset{(57),(70)}{=} p^2 \cdot \chi(\widetilde{\mathcal{N}}_{\mathbb{F}}, \mathcal{O}_{\mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}}(x) \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}} \mathcal{O}_{\mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}}(y)) + \chi(\widetilde{\mathcal{N}}_{\mathbb{F}}, \mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}} \mathcal{O}_{\mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}}) \\ &\overset{(56),(59)}{=} p^2 \cdot \chi\left(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}_{\mathcal{N}}\left(\overline{x_0}^{-1} \cdot L^\flat\right) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{N}_{\mathbb{F}}}\right) - 1.\end{aligned}$$

$\square$

6.8. **Intersections on $\mathcal{M}^{\mathrm{VF}}$.** The ideas of the computations in this part is similar to §6.7. Let

$$\left(X_1^{\mathrm{VF}} \xrightarrow{x_0^{\mathrm{V}}} X_1'^{\mathrm{VF}}, \left(\rho_1^{\mathrm{VF}}, \rho_1'^{\mathrm{VF}}\right)\right), \quad \left(X_2^{\mathrm{VF}} \xrightarrow{x_0^{\mathrm{F}}} X_2'^{\mathrm{VF}}, \left(\rho_2^{\mathrm{VF}}, \rho_2'^{\mathrm{VF}}\right)\right) \tag{71}$$

be the base change of the universal object (38) over $\mathcal{M}$ to $\mathcal{M}^{\mathrm{VF}}$. By the equations of $\mathcal{M}^{\mathrm{VF}}$ in Proposition 4.11.1, the morphism x_0^{F} is isomorphic to the Frobenius morphism, while x_0^{V} is isomorphic to the Verschiebung morphism. Hence there exist two isomorphisms $\iota_2^{\mathrm{F}} : X_2^{\mathrm{VF},(p)} \to X_2'^{\mathrm{VF}}$ and $\iota_1^{\mathrm{V}} : X_1'^{\mathrm{VF},(p)} \to X_1^{\mathrm{VF}}$ over $\mathcal{M}^{\mathrm{VF}}$ such that the following diagrams commute,

$$\begin{array}{ccc} & & X_2^{\mathrm{VF},(p)} \\ & \overset{\mathrm{F}}{\nearrow} & \\ X_2^{\mathrm{VF}} & & \downarrow \iota_2^{\mathrm{F}} \\ & \underset{x_0^{\mathrm{F}}}{\searrow} & \\ & & X_2'^{\mathrm{VF}} \end{array} \qquad \begin{array}{ccc} X_1'^{\mathrm{VF},(p)} & & \\ & \overset{\mathrm{V}}{\searrow} & \\ \iota_1^{\mathrm{V}} \downarrow & & X_1'^{\mathrm{VF}} \\ & \underset{x_0^{\mathrm{V}}}{\nearrow} & \\ X_1^{\mathrm{VF}} & & \end{array}$$

here we use F, V to represent the relative Frobenius and Verschiebung morphism.

For a morphism $g : S \to \mathcal{M}^{\mathrm{VF}}$. Let $(X_{1,S}'^{\mathrm{VF}}, \rho_{1,S}'^{\mathrm{VF}})$ and $(X_{2,S}^{\mathrm{VF}}, \rho_{2,S}^{\mathrm{VF}})$ be the base change of the p-divisible groups $(X_1'^{\mathrm{VF}}, \rho_1'^{\mathrm{VF}})$ and $(X_2^{\mathrm{VF}}, \rho_{2,S}^{\mathrm{VF}})$ to S through the morphism g. Then we get a map $\mathcal{M}^{\mathrm{VF}}(S) \to \mathcal{N}_{\mathbb{F}}(S) : g \mapsto \left(\left(X_{1,S}'^{\mathrm{VF}}, \rho_{1,S}'^{\mathrm{VF}}\right), \left(X_2^{\mathrm{VF}}, \rho_{2,S}^{\mathrm{VF}}\right)\right)$. Therefore we obtain a morphism between formal schemes $\pi^{\mathrm{VF}} : \mathcal{M}^{\mathrm{VF}} \to \mathcal{N}_{\mathbb{F}}$. Using the open cover we fixed in §4.11 and Lemma 4.6.4, we have the following isomorphism of formal schemes over $\mathcal{N}_{\mathbb{F}}$:

$$\iota^{\mathrm{VF}} : \mathcal{M}^{\mathrm{VF}} \xrightarrow{\sim} \widetilde{\mathcal{N}}_{\mathbb{F}}. \tag{72}$$

Denote by $\mathcal{M}_{\circ}^{\mathrm{VF}} = \left(\iota^{\mathrm{VF}}\right)^* \left(\widetilde{\mathcal{N}}_{\mathbb{F}}^{\circ}\right)$, $\mathcal{M}_{\bullet}^{\mathrm{FV}} = \left(\iota^{\mathrm{VF}}\right)^* \left(\widetilde{\mathcal{N}}_{\mathbb{F}}^{\bullet}\right)$. Then $\mathcal{M}_{\circ}^{\mathrm{VF}} \subset \mathcal{M}^{-}$, $\mathcal{M}_{\bullet}^{\mathrm{VF}} \subset \mathcal{M}^{+}$ and $\{\mathcal{M}_{\circ}^{\mathrm{VF}}, \mathcal{M}_{\bullet}^{\mathrm{VF}}\}$ gives an open cover of $\mathcal{M}^{\mathrm{VF}}$.

Recall that we defined $\mathrm{Fr} : \mathcal{N}_{0,\mathbb{F}} \to \mathcal{N}_{0,\mathbb{F}}$ as the relative Frobenius morphism of $\mathcal{N}_{0,\mathbb{F}}$ in §4.1. Denote by $p_{\circ}^{\mathrm{VF}} : \mathcal{M}_{\circ}^{\mathrm{VF}} \to \mathcal{N}_{\mathbb{F}}$ the restriction of the morphism $p_{-} : \mathcal{M}^{-} \to \mathcal{N}$ (cf. (39)) to $\mathcal{M}_{\circ}^{\mathrm{VF}}$. Denote by $p_{\bullet}^{\mathrm{VF}} : \mathcal{M}_{\bullet}^{\mathrm{VF}} \to \mathcal{N}_{\mathbb{F}}$ the restriction of the morphism $p_{+} : \mathcal{M}^{+} \to \mathcal{N}$ (cf. (39)) to $\mathcal{M}_{\bullet}^{\mathrm{VF}}$. Then we have

$$p_{\circ}^{\mathrm{VF}} = (\mathrm{Id} \times \mathrm{Fr}) \circ \pi^{\mathrm{VF}}, \quad p_{\bullet}^{\mathrm{VF}} = (\mathrm{Fr} \times \mathrm{Id}) \circ \pi^{\mathrm{VF}}. \tag{73}$$

Let $x \in \mathbb{B}$ be a nonzero element such that $\nu_p(q(x)) \geqslant 0$. By the moduli interpretation of the divisor $\mathcal{Z}(x)$ in Lemma 4.13.2, we have

$$\mathcal{Z}(x) \cap \mathcal{M}_{\circ}^{\mathrm{VF}} = \left(\pi^{\mathrm{VF}}\right)^* (\mathrm{Id} \times \mathrm{Fr})^* \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x'), \ \mathcal{Z}(x) \cap \mathcal{M}_{\bullet}^{\mathrm{FV}} = \left(\pi^{\mathrm{FV}}\right)^* (\mathrm{Fr} \times \mathrm{Id})^* \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x), \tag{74}$$

Notice that the quasi-isogeny $\overline{x}_0 \cdot x' = x \cdot \overline{x}_0 : X_1'^{\mathrm{FV}} \dashrightarrow X_2^{\mathrm{FV}}$ lifts to an isogeny over $\mathcal{Z}(x) \cap \mathcal{M}^{\mathrm{VF}}$. Therefore

$$\mathcal{Z}(x) \cap \mathcal{M}_{\circ}^{\mathrm{FV}} \subset \left(\pi^{\mathrm{VF}}\right)^* \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(\overline{x}_0 \cdot x'), \ \mathcal{Z}(x) \cap \mathcal{M}_{\bullet}^{\mathrm{FV}} \subset \left(\pi^{\mathrm{VF}}\right)^* \mathcal{Z}_{\mathcal{N}_{\mathbb{F}}}(x \cdot \overline{x}_0). \tag{75}$$

Lemma 6.8.1. *Let $x \in \mathbb{B}$ be a nonzero element such that $\nu_p(q(x)) \geqslant 0$. Then we have the following identity of effective Cartier divisors on $\mathcal{M}^{\mathrm{VF}}$:*

$$\mathcal{Z}(x) \cap \mathcal{M}^{\mathrm{VF}} = \left(\iota^{\mathrm{VF}}\right)^* \left(p \cdot \mathcal{Z}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}(x_0^{-1} \cdot x) + \mathrm{Exc}_{\widetilde{\mathcal{N}}_{\mathbb{F}}}\right).$$

Proof. The proof is almost identical to that of Lemma 6.7.1, so we omit it. ☐

Corollary 6.8.2. *Let $L^\flat \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$\chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}^{\mathrm{VF}}}) = p^2 \cdot \chi\left(\mathcal{N}, {}^{\mathbb{L}}\mathcal{Z}_{\mathcal{N}}\left(x_0^{-1} \cdot L^\flat\right) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}}} \mathcal{O}_{\mathcal{N}_{\mathbb{F}}}\right) - 1.$$

Proof. The proof is almost identical to that of Corollary 6.7.2, so we omit it. ☐

7. Proof of the main theorem

7.1. **Difference formulas combined.**

Lemma 7.1.1. *Let* $L^\flat \subset \mathbb{B}$ *be a* $\mathbb{Z}_p$*-lattice of rank* 2. *Let* $x \in \mathbb{B}$ *be a nonzero element such that* $x \perp L^\flat$ *and* $\nu_p(q(x)) \geqslant \max\{\max(L^\flat), 2\}$. *Then we have*

$$\chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{M}^{\mathrm{FF}}}) = \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{M}^{\mathrm{VV}}}) = \partial\mathrm{Den}\left(\langle x\rangle[-1] \oplus H_2^+, L^\flat\right), \tag{76}$$

$$\chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{M}^{\mathrm{FV}}}) = \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{M}^{\mathrm{VF}}}) = \partial\mathrm{Den}\left(\langle x\rangle[-1] \oplus H_2^+[p], L^\flat\right). \tag{77}$$

Proof. The formula (76) is proved by combining Lemma 6.5.1, Lemma 6.6.1 and Lemma 6.1.2. Let's now give the proof of the formula (77). By Corollary 6.7.2, 6.8.2 and Lemma 6.1.2 again, we have

$$\chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{M}^{\mathrm{FV}}}) = \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{M}^{\mathrm{VF}}}) = p^2 \cdot \partial\mathrm{Den}\left(\langle x\rangle[-p^{-1}] \oplus H_2^+, L^\flat[p^{-1}]\right) - 1.$$

By Lemma 3.4.1, we have

$$\partial\mathrm{Den}\left(\langle x\rangle[-1] \oplus H_2^+[p], L^\flat\right) = p^2 \cdot \partial\mathrm{Den}\left(\langle x\rangle[-p^{-1}] \oplus H_2^+, L^\flat[p^{-1}]\right) - 1.$$

Therefore (77) is true. □

Corollary 7.1.2. *Let* $L^\flat \subset \mathbb{B}$ *be a* $\mathbb{Z}_p$*-lattice of rank* 2. *Let* $x \in \mathbb{B}$ *be a nonzero element such that* $x \perp L^\flat$ *and* $\nu_p(q(x)) \geqslant \max\{\max(L^\flat), 2\}$. *Then we have*

$$\mathrm{Int}^{\mathcal{Z}}\left(L^\flat \oplus \langle x\rangle\right) - \mathrm{Int}^{\mathcal{Z}}\left(L^\flat \oplus \langle p^{-1}x\rangle\right) = \partial\mathrm{Den}\left(H_0(p), L^\flat \oplus \langle x\rangle\right) - \partial\mathrm{Den}\left(H_0(p), L^\flat \oplus \langle p^{-1}x\rangle\right).$$

$$\mathrm{Int}^{\mathcal{Y}}\left(L^\flat \oplus \langle x\rangle\right) - \mathrm{Int}^{\mathcal{Y}}\left(L^\flat \oplus \langle p^{-1}x\rangle\right) = \partial\mathrm{Den}\left(H_0(p)^\vee, L^\flat \oplus \langle x\rangle\right) - \partial\mathrm{Den}\left(H_0(p)^\vee, L^\flat \oplus \langle p^{-1}x\rangle\right).$$

Proof. For the intersection number of $\mathcal{Z}$-cycles, we have

$$\begin{aligned}\mathrm{Int}^{\mathcal{Z}}\left(L^\flat \oplus \langle x\rangle\right) - \mathrm{Int}^{\mathcal{Z}}\left(L^\flat \oplus \langle p^{-1}x\rangle\right) &= 2 \cdot \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{M}^{\mathrm{FF}}}) + 2 \cdot \chi(\mathcal{M}, {}^{\mathbb{L}}\mathcal{Z}(L^\flat) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{M}^{\mathrm{FV}}}) \\ &\overset{(76),(77)}{=} 2 \cdot \partial\mathrm{Den}\left(\langle x\rangle[-1] \oplus H_2^+, L^\flat\right) + 2 \cdot \partial\mathrm{Den}\left(\langle x\rangle[-1] \oplus H_2^+[p], L^\flat\right).\end{aligned}$$

On the analytic side, we have the following formula by Lemma 3.3.3,

$$\begin{aligned}\partial\mathrm{Den}\left(H_0(p), L^\flat \oplus \langle x\rangle\right) &- \partial\mathrm{Den}\left(H_0(p), L^\flat \oplus \langle p^{-1}x\rangle\right) \\ &= 2 \cdot \partial\mathrm{Den}\left(\langle x\rangle[-1] \oplus H_2^+, L^\flat\right) + 2 \cdot \partial\mathrm{Den}\left(\langle x\rangle[-1] \oplus H_2^+[p], L^\flat\right).\end{aligned}$$

Hence the first formula follows from the above two identities.

For the intersection number of $\mathcal{Y}$-cycles. By the identity $\mathcal{Y}(H) = \left(\iota^{\mathcal{M}}\right)^* \mathcal{Z}(x_0 \cdot H)$, we have

$$\begin{aligned}&\mathrm{Int}^{\mathcal{Y}}\left(L^\flat \oplus \langle x\rangle\right) - \mathrm{Int}^{\mathcal{Y}}\left(L^\flat \oplus \langle p^{-1}x\rangle\right) \\ &= \mathrm{Int}^{\mathcal{Z}}\left(L^\flat[q(x_0)] \oplus \langle x\rangle[q(x_0)]\right) - \mathrm{Int}^{\mathcal{Z}}\left(L^\flat[q(x_0)] \oplus \langle p^{-1}x\rangle[q(x_0)]\right) \\ &= \partial\mathrm{Den}\left(H_0(p), L^\flat[q(x_0)] \oplus \langle x\rangle[q(x_0)]\right) - \partial\mathrm{Den}\left(H_0(p), L^\flat[q(x_0)] \oplus \langle p^{-1}x\rangle[q(x_0)]\right) \\ &= \partial\mathrm{Den}\left(H_0(p)^\vee, L^\flat \oplus \langle x\rangle\right) - \partial\mathrm{Den}\left(H_0(p)^\vee, L^\flat \oplus \langle p^{-1}x\rangle\right).\end{aligned}$$

Here the last identity is proved by Lemma 3.4.2. □

7.2. **Base cases: the geometric side.** Let $L \subset \mathbb{B}$ be a quadratic lattice of rank 3 such that $\min(L) = 0$. Let $x \in L$ be an element such that $\nu_p(q(x)) = 0$. By Lemma 5.5.2, we have the following equality of Cartier divisors on $\mathcal{M}$,

$$\mathcal{Z}(x) = \mathrm{Exc}_{\mathcal{M}} + \mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x),$$

where the divisor $\mathcal{N}_0^{\mathrm{I}+}(x_0 \cdot x)$ is isomorphic to the blow up $\widetilde{\mathcal{N}}_0(x_0 \cdot x)$ of the formal scheme $\mathcal{N}_0(x_0 \cdot x)$ along its unique closed $\mathbb{F}$-point. For simplicity, let $z = x_0 \cdot x$. We fix an isomorphism $\iota_z : \mathcal{N}_0^{\mathrm{I}+}(z) \xrightarrow{\sim} \widetilde{\mathcal{N}}_0(z)$. Let p_z be the following composition morphism

$$p_z : \mathcal{N}_0^{\mathrm{I}+}(z) \xrightarrow{\iota_z} \widetilde{\mathcal{N}}_0(z) \xrightarrow{\pi_z} \mathcal{N}_0(z) \hookrightarrow \mathcal{N}.$$

Recall that we use $\mathrm{Exc}_z^{\mathrm{I}+}$ to denote the exceptional divisor on $\mathcal{N}_0^{\mathrm{I}+}(z)$.

Lemma 7.2.1. *Let $y \in \mathbb{B}$ be an element such that x, y are linearly independent. Then we have*

$$\chi(\mathcal{N}_0^{\mathrm{I}+}(z), \mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(z)\cap\mathcal{Z}(y)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(z)}} \mathcal{O}_{\mathrm{Exc}_z^{\mathrm{I}+}}) = -1. \tag{78}$$

Proof. By definition we know that $\mathrm{Exc}_z^{\mathrm{I}+} = \mathrm{Exc}_{\mathcal{M}} \cap \mathcal{N}_0^{\mathrm{I}+}(z)$. As a divisor on $\mathrm{Exc}_{\mathcal{M}}$, we have

$$[\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(z)}] = [\mathcal{O}_{\mathrm{Exc}_z^{\mathrm{I}+}}] = \mathcal{O}(1,0)$$

by Corollary 5.3.2 (b). Then we have

$$\begin{aligned}\chi(\mathcal{N}_0^{\mathrm{I}+}(z), \mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(z)\cap\mathcal{Z}(y)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(z)}} \mathcal{O}_{\mathrm{Exc}_z^{\mathrm{I}+}}) &= \chi(\mathcal{M}, \mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(z)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}) \\ &= \chi(\mathrm{Exc}_{\mathcal{M}}, \mathcal{O}_{\mathrm{Exc}_z^{\mathrm{I}+}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}} \mathcal{O}_{\mathcal{Z}(y)\cap\mathrm{Exc}_{\mathcal{M}}}) \\ &= \mathcal{O}(1,0)\cdot\mathcal{O}(-1,-1) = -1.\end{aligned}$$

□

Let $y \in \mathbb{B}$ be an element such that it is linearly independent from x and $\nu_p(q(y)) = 1$. Denote by $z' = x_0 \cdot y$. By the moduli interpretation of the special divisor $\mathcal{Z}(y)$, we have

$$(p_z)^*\left(\mathcal{Z}_{\mathcal{N}}(p^{-1}z')\right) = (\pi_z\iota_z)^*\left(\mathcal{N}_0(z) \cap \mathcal{Z}_{\mathcal{N}}(p^{-1}z')\right) \subset \mathcal{N}_0^{\mathrm{I}+}(z) \cap \mathcal{Z}(y).$$

Notice that $\nu_p(q(p^{-1}z')) = 0$, hence the intersection $\mathcal{N}_0(z) \cap \mathcal{Z}_{\mathcal{N}}(p^{-1}z')$ is a regular divisor on $\mathcal{N}_0(z)$ which is isomorphic to $\mathrm{Spf}\, W_0$ by [GK93, (5.10)], where W_0 is the integer ring of a ramified quadratic extension of $\mathbb{Q}_p$. Denote this divisor by $\mathcal{C}(y)$. Let $\widetilde{\mathcal{C}}(y)$ be the strict transform of the divisor $\mathcal{C}(y)$ under the blow up morphism $\pi_z : \widetilde{\mathcal{N}}_0(z) \to \mathcal{N}_0(z)$. Let $p_{z,1}, p_{z,2} : \mathcal{N}_0^{\mathrm{I}+}(z) \to \mathcal{N}_0$ be the composition of p_z with the two projections from $\mathcal{N} \simeq \mathcal{N}_0 \times_W \mathcal{N}_0$ to $\mathcal{N}_0$.

Lemma 7.2.2. *Let $y \in \mathbb{B}$ be an element such that it is perpendicular to x and $\nu_p(q(y)) = 0$ or 1. Then*

$$\mathcal{N}_0^{\mathrm{I}+}(z) \cap \mathcal{Z}(y) = \begin{cases} \iota_z^*\left(\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(z)}\right), & \text{if } \nu_p(q(y)) = 0; \\ \iota_z^*\left(2\cdot\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(z)} + \widetilde{\mathcal{C}}(y)\right), & \text{if } \nu_p(q(y)) = 1. \end{cases}$$

Proof. Using Lemma 4.4.1 (b), we can construct the following open cover of the formal scheme $\widetilde{\mathcal{N}}_0(z)$:

$$\widetilde{\mathcal{N}}_0(z)^\circ = \mathrm{Spf}\, W[u][[t]]/\left(\nu p + t^2(t^{p-1} - u)\right),\ \widetilde{\mathcal{N}}_0(z)^\bullet = \mathrm{Spf}\, W[v][[t']]/\left(\nu' p + t'^2(t^{p-1} - v)\right), \tag{79}$$

where ν, ν' are invertible elements in the corresponding rings and $uv = 1$. Using Lemma 4.6.4, we can see easily that $\widetilde{\mathcal{N}}_0(z)^\circ \subset \mathcal{M}^+$ and $\widetilde{\mathcal{N}}_0(z)^\bullet \subset \mathcal{M}^-$. Let $y_1 = x^{-1}y$. Then we have $y_1 \in \mathbb{B}^0$. By the moduli interpretation of $\mathcal{Z}(y)$ in Lemma 4.13.2, we have

$$\mathcal{Z}(y) \cap \widetilde{\mathcal{N}}_0(z)^\circ = (p_{z,1})^* \mathcal{Z}_{\mathcal{N}_0}(y_1), \quad \mathcal{Z}(y) \cap \widetilde{\mathcal{N}}_0(z)^\bullet = (p_{z,2})^* \mathcal{Z}_{\mathcal{N}_0}(y_1').$$

- If $\nu_p(q(y)) = 0$, we have $\nu_p(q(y_1)) = \nu_p(q(y_1')) = 0$. The equation cutting out $\mathcal{Z}_{\mathcal{N}_0}(y_1)$ (resp. $\mathcal{Z}_{\mathcal{N}_0}(y_1')$) has the form $t + a \cdot p$ (resp. $t' + a' \cdot p$) for some $a \in \mathcal{O}_{\mathcal{N}_0}$ (resp. $a' \in \mathcal{O}_{\mathcal{N}_0}$) since $\mathcal{Z}_{\mathcal{N}_0}(y_1)$ (resp. $\mathcal{Z}_{\mathcal{N}_0}(y_1')$) is isomorphic to $\mathrm{Spf}\, W$ by Remark 4.3.4. Using the open cover (79), the equation of $\mathcal{Z}(y)$ on $\widetilde{\mathcal{N}}_0(z)^\circ$ (resp. $\widetilde{\mathcal{N}}_0(z)^\bullet$) is given by $t \times$(a unit element in $\mathcal{O}_{\widetilde{\mathcal{N}}_0(z)^\circ}$) (resp. $t' \times$(a unit element in $\mathcal{O}_{\widetilde{\mathcal{N}}_0(z)^\bullet}$)). Therefore we have
$$\mathcal{N}_0^{\mathrm{I}+}(z) \cap \mathcal{Z}(y) = \iota_z^* \left(\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(z)}\right).$$
- If $\nu_p(q(y)) = 1$, we have $\nu_p(q(y_1)) = \nu_p(q(y_1')) = 1$. The equation cutting out $\mathcal{Z}_{\mathcal{N}_0}(y_1)$ (resp. $\mathcal{Z}_{\mathcal{N}_0}(y_1')$) has the form $p + t^2 \cdot f(t)$ where $f(t) \in W[[t]]$ (resp. $p + t'^2 g(t')$ where $g(t') \in W[[t']]$) since $\mathcal{Z}_{\mathcal{N}_0}(y_1)$ (resp. $\mathcal{Z}_{\mathcal{N}_0}(y_1')$) is a regular formal scheme but not formally smooth over W by Remark 4.3.4. Using the open cover (79), the equation of $\mathcal{Z}(y)$ on $\widetilde{\mathcal{N}}_0(z)^\circ$ (resp. $\widetilde{\mathcal{N}}_0(z)^\bullet$) is given by $t^2 \cdot \left(\nu^{-1}(u - t^{p-1}) + f(t)\right)$ (resp. $t'^2 \cdot \left(\nu'^{-1}(v - t'^{p-1}) + g(t')\right)$). Therefore the multiplicity of $\mathrm{Exc}_z^{\mathrm{I}+}$ in the divisor $\mathcal{N}_0^{\mathrm{I}+}(z) \cap \mathcal{Z}(y)$ is exactly 2. Since every horizontal divisor in $\mathcal{N}_0^{\mathrm{I}+}(z) \cap \mathcal{Z}(y)$ has intersection number 1 with $\mathrm{Exc}_z^{\mathrm{I}+}$, we conclude that there is only one horizontal divisor by Lemma 7.2.1. Notice that $\widetilde{\mathcal{C}}(y) \subset \mathcal{N}_0^{\mathrm{I}+}(z) \cap \mathcal{Z}(y)$ is a horizontal divisor. Therefore
$$\mathcal{N}_0^{\mathrm{I}+}(z) \cap \mathcal{Z}(y) = \iota_z^* \left(2 \cdot \mathrm{Exc}_{\widetilde{\mathcal{N}}_0(z)} + \widetilde{\mathcal{C}}(y)\right).$$

□

Corollary 7.2.3. *Let $L \subset \mathbb{B}$ be a lattice of rank 3 whose Gross–Keating invariant $\mathrm{GK}(L) = (0,0,1)$ or $(0,1,1)$. Then*

$$\mathrm{Int}^{\mathcal{Z}}(L) = \begin{cases} -1, & \text{if } \mathrm{GK}(L) = (0,0,1); \\ 0, & \text{if } \mathrm{GK}(L) = (0,1,1). \end{cases}$$

Similarly, we have

$$\mathrm{Int}^{\mathcal{Y}}(L) = \begin{cases} -1, & \text{if } \mathrm{GK}(L) = (-1,-1,0); \\ 0, & \text{if } \mathrm{GK}(L) = (-1,0,0). \end{cases}$$

Proof. Note that the statement for $\mathrm{Int}^{\mathcal{Y}}(L)$ follows from $\mathrm{Int}^{\mathcal{Z}}(L)$ via (33). Now we prove the statement for $\mathrm{Int}^{\mathcal{Z}}(L)$.

Let x, y_1, y_2 be an orthogonal basis of the lattice L where $\nu_p(q(x)) = 0$ and $\nu_p(q(y_2)) = 1$. Let $z = x_0 \cdot x$. We have

$$\begin{aligned}\mathrm{Int}^{\mathcal{Z}}(L) = \chi\left(\mathcal{M}, \mathcal{O}_{\mathcal{Z}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y_2)}\right) &= \chi\left(\mathcal{M}, \mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(z)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y_2)}\right) \\ &= \chi\left(\mathcal{N}_0^{\mathrm{I}+}(z), \mathcal{O}_{\mathcal{Z}(y_1) \cap \mathcal{N}_0^{\mathrm{I}+}(z)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}(y_2) \cap \mathcal{N}_0^{\mathrm{I}+}(z)}\right).\end{aligned}$$

In the following, we use $(\cdot,\cdot)_{\widetilde{\mathcal{N}}_0(z)}$ (resp. $(\cdot,\cdot)_{\mathcal{N}_0(z)}$) to denote the intersection pairing of divisors on $\widetilde{\mathcal{N}}_0(z)$ (resp. $\mathcal{N}_0(z)$).

- $\nu_p(q(y_1)) = 0$. By Lemma 7.2.2, we have
$$\mathrm{Int}^{\mathcal{Z}}(L) = \left(\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(z)}, 2\cdot \mathrm{Exc}_{\widetilde{\mathcal{N}}_0(z)} + \widetilde{\mathcal{C}}(y_2)\right)_{\widetilde{\mathcal{N}}_0(z)} = -2+1=-1.$$
- $\nu_p(q(y_1)) = 1$. By the construction of $\widetilde{\mathcal{C}}(y_i)$ where $i=1,2$, we have
$$\mathcal{C}(y_i) \simeq \mathcal{Z}_{\mathcal{N}_0}(p^{-1}x_0\cdot y_i). \tag{80}$$
By Lemma 7.2.2, we have
$$\mathcal{N}_0^{\mathrm{I}+}(z)\cap \mathcal{Z}(y_i) = \iota_z^*\left(\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(z)} + \pi_z^*\mathcal{C}(y_i)\right).$$
By Lemma 4.5.1, we have
$$\begin{aligned}\mathrm{Int}^{\mathcal{Z}}(L) &= \left(\mathrm{Exc}_{\widetilde{\mathcal{N}}_0(z)} + \pi^*\mathcal{C}(y_1), \mathrm{Exc}_{\widetilde{\mathcal{N}}_0(z)} + \pi^*\mathcal{C}(y_2)\right)_{\widetilde{\mathcal{N}}_0(z)}\\ &= -1+0+0+(\pi^*\mathcal{C}(y_1),\pi^*\mathcal{C}(y_2))_{\widetilde{\mathcal{N}}_0(z)} = -1+(\mathcal{C}(y_1),\mathcal{C}(y_2))_{\mathcal{N}_0(z)}.\end{aligned}$$
Notice that $p^{-1}x_0\cdot y_1, p^{-1}x_0\cdot y_2, z$ span an orthogonal basis of a rank 3 lattice in $\mathbb{B}$ whose Gross–Keating invariant is $(0,0,1)$. By (80) and the fact that $\mathcal{N}_0(z) = \mathcal{Z}_{\mathcal{N}}(z)$, we have $(\mathcal{C}(y_1),\mathcal{C}(y_2))_{\mathcal{N}_0(z)} = 1$ by [GK93, Proposition 5.4]. Therefore $\mathrm{Int}^{\mathcal{Z}}(L)=0$.

□

7.3. **Proof of Theorem 5.6.7.** Theorem 5.6.7 states that for a $\mathbb{Z}_p$-lattice $L\subset\mathbb{B}$ of rank 3, we have
$$\mathrm{Int}^{\mathcal{Z}}(L) = \partial\mathrm{Den}(H_0(p), L). \tag{81}$$
and
$$\mathrm{Int}^{\mathcal{Y}}(L) = \partial\mathrm{Den}(H_0(p)^\vee, L) - 1. \tag{82}$$

We first prove (81). Notice that this is automatically true if L is not integral because both sides are 0. Now we assume that the quadratic lattice L is integral. Let (a_1,a_2,a_3) be the Gross–Keating invariant of L where $0\leqslant a_1\leqslant a_2\leqslant a_3$ are integers. Let $n(L)=a_1+a_2+a_3$. We prove the equality (81) by induction on n.

1. Base cases: $n(L)=1$ or 2. The equality (81) is true by combining Lemma 3.5.1 and Corollary 7.2.3.
2. Induction step: If the equality (81) is true for all quadratic lattices L such that $n(L)\leqslant k$ where $k\geqslant 2$ is an integer. Let L' be a quadratic lattice such that $n(L')=k+1$. Let x_1,x_2,x_3 be an orthogonal basis of L' such that $\nu_p(q(x_1))\leqslant\nu_p(q(x_2))\leqslant\nu_p(q(x_3))$. Then $\nu_p(q(x_3))\geqslant 2$. Let $L^\flat = \mathbb{Z}_p\cdot x_1\oplus\mathbb{Z}_p\cdot x_2$ and $L = L^\flat\oplus\mathbb{Z}_p\cdot p^{-1}x_3$. Then we have $n(L)=k-1<k$. The equality (81) holds by combining $\mathrm{Int}^{\mathcal{Z}}(L)=\partial\mathrm{Den}(H_0(p),L)$ and Corollary 7.1.2.

Therefore we can conclude that the equality (81) is true for all lattices $L\subset\mathbb{B}$ of rank 3.

The proof of (82) is similar so we omit it. □

Part 3. Applications

8. A conjecture of Kudla–Rapoport

8.1. **CM cycles on $\mathcal{N}_0(x_0)$.** Roughly speaking, the formal scheme $\mathcal{N}_0(x_0)$ parameterizes cyclic isogenies deforming x_0. Since $\nu_p(q(x_0)) = 1$, we have $\mathcal{N}_0(x_0) = \mathcal{Z}_{\mathcal{N}}(x_0)$ by Lemma 4.4.1. Let

$$\left(X^{\mathrm{univ}} \xrightarrow{\pi} X'^{\mathrm{univ}}, (\rho^{\mathrm{univ}}, \rho'^{\mathrm{univ}})\right)$$

be the universal deformation of the isogeny x_0 over $\mathcal{N}_0(x_0)$. Let $\mathbb{B}^0$ be the subset of $\mathbb{B}$ consisting of trace 0 elements. Let S be the subset of $H_0(p)$ consisting of trace 0 elements.

Definition 8.1.1. Let $H \subset \mathbb{B}^0$ be a subset. Define the special cycle $\mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(H) \subset \mathcal{N}_0(x_0)$ to be the closed formal subscheme cut out by the conditions,

$$\rho^{\mathrm{univ}} \circ x \circ (\rho^{\mathrm{univ}})^{-1} \in \mathrm{End}(X^{\mathrm{univ}});$$
$$\rho'^{\mathrm{univ}} \circ x' \circ (\rho'^{\mathrm{univ}})^{-1} \in \mathrm{End}(X'^{\mathrm{univ}}).$$

for all $x \in H$.

For a given element $x \in \mathbb{B}^0$, the special cycle $\mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(x)$ is not a divisor and has embedded components. Let $\mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(x)^\flat$ be the associated divisor of $\mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(x)$ following the notion of Kudla and Rapoport [KR00, §4]. In the current situation, let $f_x, f_{x'} \in \mathcal{O}_{\mathcal{N}_0(x_0)}$ be the two elements cutting out the closed formal subschemes over which x and x' are isogenies respectively. Let g_x be the greatest common divisor of f_x and $f_{x'}$, then the associated divisor of $\mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(x)$ is defined by the element g_x.

In 2006, Michael Rapoport gave on talk titled "Some remarks on special cycles on Shimura curves". In that talk, he explained his conjecture with Steven Kudla as follows. Let $S \subset H_0(p)$ be the sublattice of trace 0 elements. Note that if $v \in H_0(p)$ is a vector with $q(v) = 1$, then S is isometric to the perpendicular lattice of v in $H_0(p)$. Let

$$\partial\mathrm{Den}\,(S, M) := -\,2 \cdot \frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \frac{\mathrm{Den}\,(X, S, M)}{\mathrm{Den}\,\big(S, H_1 \oplus H_1^+[p]\big)},$$

where H_1 is the perpendicular lattice of v in H_2^+.

Conjecture 8.1.2. Let $x, y \in \mathbb{B}^0$ be two linearly independent vectors. Let M be the $\mathbb{Z}_p$-lattice spanned by x and y. Then

$$\chi\left(\mathcal{N}_0(x_0), \mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(x)^\flat} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(x_0)}} \mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(y)^\flat}\right) = \partial\mathrm{Den}(S, M) + 1.$$

In the following paragraphs, we will reformulate and then confirm this conjecture.

8.2. **Geometric cancellation law.** Let $1 \in \mathbb{B}$ be the identity element. By the definition of the special cycles on $\mathcal{N}(x_0)$, there exists an isomorphism $\iota_0 : \mathcal{N}_0(x_0) \simeq \mathcal{Z}_{\mathcal{N}(x_0)}(1)$. For a subset $H \subset \mathbb{B}^0$, let $H' = \{1\} \cup H$. We have

$$\mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(H) = (\iota_0)^*(\mathcal{Z}_{\mathcal{N}(x_0)}(H'))$$

Recall that we denote by $\widetilde{\mathcal{N}}_0(x_0)$ the blow up of the formal scheme $\mathcal{N}_0(x_0)$ along its unique closed $\mathbb{F}$-point. The isomorphism ι_0 induces a closed immersion $\widetilde{\mathcal{N}}_0(x_0) \to \mathcal{Z}(1) \subset \mathcal{M}$. By Lemma 5.5.2, this closed immersion induces an isomorphism $\widetilde{\mathcal{N}}_0(x_0) \xrightarrow{\sim} \mathcal{N}_0^{\mathrm{I}+}(1) = \widetilde{\mathcal{D}}(1) = \widetilde{\mathcal{Z}}(1)$. We still denote this isomorphism by ι_0.

Definition 8.2.1. Let $H \subset \mathbb{B}^0$ be a subset. Define the special cycle $\mathcal{Z}^{\mathrm{CM}}(H) \subset \widetilde{\mathcal{N}}_0(x_0)$ to be

$$\mathcal{Z}^{\mathrm{CM}}(H) := \mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(H) \times_{\mathcal{N}_0(x_0)} \widetilde{\mathcal{N}}_0(x_0).$$

For a single element $x \in \mathbb{B}^0$, denote by $\mathcal{Z}^{\mathrm{CM}}(x)$ the cycle $\mathcal{Z}^{\mathrm{CM}}(\{x\})$.

Recall that we use $\left(X_{1,\mathcal{M}} \xrightarrow{(x_0)_{1,\mathcal{M}}} X'_{1,\mathcal{M}}, \left(\rho_{1,\mathcal{M}}, \rho'_{1,\mathcal{M}}\right)\right)$, $\left(X_{2,\mathcal{M}} \xrightarrow{(x_0)_{2,\mathcal{M}}} X'_{2,\mathcal{M}}, \left(\rho_{2,\mathcal{M}}, \rho'_{2,\mathcal{M}}\right)\right)$ to denote the universal object over $\mathcal{M}$. Over the formal scheme $\widetilde{\mathcal{N}}_0(x_0) \simeq \mathcal{N}_0^{\mathrm{I}+}(1) \subset \mathcal{M}$, we have the following commutative diagram

$$\begin{array}{ccc} X_{1,\mathcal{M}} & \xrightarrow[\sim]{1} & X_{2,\mathcal{M}} \\ {\scriptstyle x_{0,1}^{\mathrm{univ}}}\downarrow & & \downarrow{\scriptstyle x_{0,2}^{\mathrm{univ}}} \\ X'_{1,\mathcal{M}} & \xrightarrow[\sim]{1} & X'_{2,\mathcal{M}}, \end{array} \tag{83}$$

We can identify $X_{1,\mathcal{M}}$ with $X_{2,\mathcal{M}}$ (resp. $X'_{1,\mathcal{M}}$ with $X'_{2,\mathcal{M}}$) using the diagram (83). Comparing the definition of $\mathcal{Z}(H)$ in Definition 4.13.1 and the definition of $\mathcal{Z}^{\mathrm{CM}}(H)$ in Definition 8.2.1, we get

Lemma 8.2.2. *Let $H \subset \mathbb{B}^0$ be a subset. Then*

$$\mathcal{Z}^{\mathrm{CM}}(H) = (\iota_0)^* \left(\mathcal{N}_0^{\mathrm{I}+}(x_0) \cap \mathcal{Z}(H)\right).$$

Corollary 8.2.3. *Let $x \in \mathbb{B}^0$. The CM cycle $\mathcal{Z}^{\mathrm{CM}}(x)$ is an effective Cartier divisor on the formal scheme $\widetilde{\mathcal{N}}_0(x_0)$.*

Proof. By Lemma 8.2.2, we know that $\mathcal{Z}^{\mathrm{CM}}(x) = \left(\mathcal{Z}(x) \cap \mathcal{N}_0^{\mathrm{I}+}(1)\right) \times_{\mathcal{N}_0^{\mathrm{I}+}(1),\iota_0} \widetilde{\mathcal{N}}_0(x_0)$. Lemma 4.13.2 shows that $\mathcal{Z}(x)$ is a divisor on $\mathcal{M}$, hence $\mathcal{Z}(x) \cap \mathcal{N}_0^{\mathrm{I}+}(1)$ is a divisor on $\mathcal{N}_0^{\mathrm{I}+}(1)$. Therefore $\mathcal{Z}^{\mathrm{CM}}(x)$ is an effective Cartier divisor on the formal scheme $\widetilde{\mathcal{N}}_0(x_0)$. ☐

Lemma 8.2.4. *Let $x \in \mathbb{B}^0$. Let $\pi_{x_0} : \widetilde{\mathcal{N}}_0(x_0) \to \mathcal{N}_0(x_0)$ be the blow up morphism with exceptional divisor* Exc_{x_0}. *Then*

$$\mathcal{Z}^{\mathrm{CM}}(x) = \pi_{x_0}^* \left(\mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(x)^\flat\right) + \mathrm{Exc}_{x_0}.$$

Proof. Let $f_x, f_{x'} \in \mathcal{O}_{\mathcal{N}_0(x_0)}$ be the two elements cutting out the closed formal subschemes over which x and x' are isogenies respectively. Let g_x be the greatest common divisor of f_x and $f_{x'}$. Then $(f_x, f_{x'}) = (g_x) \cdot \mathfrak{m}_x$ where $\mathfrak{m}_x \subset \mathcal{O}_{\mathcal{N}_0(x_0)}$ is a primary ideal whose radical is the maximal ideal by the proof of [KR00, Lemma 4.2]. Therefore $\pi_{x_0}^*\mathfrak{m}_x$ is a multiple of Exc_{x_0}, i.e.,

$$\mathcal{Z}^{\mathrm{CM}}(x) = \pi_{x_0}^* \left(\mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(x)^\flat\right) + m_x \cdot \mathrm{Exc}_{x_0}. \tag{84}$$

By Lemma 7.2.1, we have

$$\chi\left(\widetilde{\mathcal{N}}_0(x_0), \mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x_0)}} \mathcal{O}_{\mathrm{Exc}_{x_0}}\right) = -1.$$

Notice that $\chi\left(\widetilde{\mathcal{N}}_0(x_0), \mathcal{O}_{\pi_{x_0}^*\left(\mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(x)^\flat\right)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x_0)}} \mathcal{O}_{\mathrm{Exc}_{x_0}}\right) = 0$, hence by (84), we get $-m_x = -1$, therefore $m_x = 1$. ☐

By the above lemma and Lemma 4.5.1 (d), we obtain the following:

Corollary 8.2.5. *Let $x, y \in \mathbb{B}^0$ be two linearly independent vectors. Then*

$$\chi\left(\mathcal{N}_0(x_0), \mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(x)^\flat} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(x_0)}} \mathcal{Z}^{\mathrm{CM}}_{\mathcal{N}_0(x_0)}(y)^\flat\right) = \chi\left(\widetilde{\mathcal{N}}_0(x_0), \mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x_0)}} \mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}(y)}\right) + 1.$$

8.3. Analytic cancellation law.

Lemma 8.3.1. *Let $M \subset \mathbb{B}^0$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$\partial\mathrm{Den}(S, M) = \partial\mathrm{Den}(H_0(p), \mathbb{Z}_p \cdot 1 \oplus M).$$

Proof. In Lemma 3.3.3, we take $L^\flat = M$ and $x = 1$. Then $\nu_p(q(x)) = 0$. The formula in this lemma corresponds to the "$n = 0$" case of Lemma 3.3.3. □

8.4. Derived CM cycles ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{CM}}(M)$.

Lemma 8.4.1. *Let $M \subset \mathbb{B}^0$ be a $\mathbb{Z}_p$-lattice of rank 2. Let $\{x, y\}$ be a basis of M, then the element $[\mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x_0)}} \mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}(y)}]$ belongs to the group $\mathrm{Gr}^2 K_0^{\mathcal{Z}^{\mathrm{CM}}(M)}(\widetilde{\mathcal{N}}_0(x_0))$ and only depends on the $\mathbb{Z}_p$-lattice M.*

Proof. Let $L = M \oplus \langle x_0 \rangle$. Notice that

$$\begin{aligned}
{}^{\mathbb{L}}\mathcal{Z}(L) &= [\mathcal{O}_{\mathcal{Z}(x_0)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{Z}(x_2)}] \\
&= [\mathcal{O}_{\mathrm{Exc}_\mathcal{M}} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{Z}(x_2)}] + [\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{Z}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{M}} \mathcal{O}_{\mathcal{Z}(y)}] \\
&= [\mathcal{O}(0,-1) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathrm{Exc}_\mathcal{M}}} \mathcal{O}(0,-1)] + [\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0) \cap \mathcal{Z}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)}} \mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0) \cap \mathcal{Z}(y)}].
\end{aligned}$$

The elements $[\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)\cap\mathcal{Z}(x)}] \in \mathrm{Gr}^1 K_0^{\mathcal{Z}(x)}(\mathcal{N}_0^{\mathrm{I}+}(x_0))$ and $[\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)\cap\mathcal{Z}(y)}] \in \mathrm{Gr}^1 K_0^{\mathcal{Z}(y)}(\mathcal{N}_0^{\mathrm{I}+}(x_0))$ are both linear combinations of (quasi-)canonical liftings of the form $[\mathcal{O}_{W_s}]$ (horizontal part) and $[\mathcal{O}_{\mathrm{Exc}^{\mathrm{I}+}_{x_0\cdot x}}]$ (vertical part) by Lemma 5.6.1. In the derived intersection $[\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)\cap\mathcal{Z}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)}} \mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)\cap\mathcal{Z}(y)}]$, the derived intersections between (quasi-)canonical liftings are proper ([Gro86], [GK93]); the derived intersection of (quasi-)canonical liftings with $[\mathcal{O}_{\mathrm{Exc}^{\mathrm{I}+}_{x_0\cdot x}}]$ is also proper; the derived intersection $[\mathcal{O}_{\mathrm{Exc}^{\mathrm{I}+}_{x_0\cdot x}} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)}} \mathcal{O}_{\mathrm{Exc}^{\mathrm{I}+}_{x_0\cdot x}}] \in \mathrm{Gr}^2 K_0^{\mathrm{Exc}^{\mathrm{I}+}_{x_0\cdot x}}(\mathcal{N}_0^{\mathrm{I}+}(x_0))$ because $\mathrm{Exc}^{\mathrm{I}+}_{x_0\cdot x} \simeq \mathbb{P}^1_\mathbb{F}$ is an integral Noetherian scheme. Therefore the element $[\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)\cap\mathcal{Z}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)}} \mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)\cap\mathcal{Z}(y)}]$ belongs to the group $\mathrm{Gr}^2 K_0^{\mathcal{Z}(L)}(\mathcal{N}_0^{\mathrm{I}+}(x_0))$.

Since ${}^{\mathbb{L}}\mathcal{Z}(L)$ is independent of the basis of L by Lemma 5.6.3, the element $[\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)\cap\mathcal{Z}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)}} \mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)\cap\mathcal{Z}(y)}] \in \mathrm{Gr}^2 K_0^{\mathcal{Z}(L)}(\mathcal{N}_0^{\mathrm{I}+}(x_0))$ is independent of basis x, y of M. Notice that we have an isomorphism $\iota_0 : \widetilde{\mathcal{N}}_0(x_0) \xrightarrow{\sim} \mathcal{N}_0^{\mathrm{I}+}(x_0)$. By Lemma 8.2.2, we have

$$[\mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x_0)}} \mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}(y)}] = (\iota_0)^* [\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)\cap\mathcal{Z}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)}} \mathcal{O}_{\mathcal{N}_0^{\mathrm{I}+}(x_0)\cap\mathcal{Z}(y)}]$$

Therefore $[\mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_0(x_0)}} \mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}(y)}] \in \mathrm{Gr}^2 K_0^{\mathcal{Z}^{\mathrm{CM}}(M)}(\widetilde{\mathcal{N}}_0(x_0))$ only depends on the $\mathbb{Z}_p$-lattice M. □

Based on Lemma 8.4.1, the following definition is reasonable.

Definition 8.4.2. Let $M \subset \mathbb{B}^0$ be a $\mathbb{Z}_p$-lattice of rank 2. Let $\{x, y\}$ be a basis of M. Define the derived CM cycle to be

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{CM}}(M) = [\mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}(x)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{Z}^{\mathrm{CM}}(y)}] \in \mathrm{Gr}^2 K_0^{\mathcal{Z}^{\mathrm{CM}}(M)}(\widetilde{\mathcal{N}}_0(x_0)).$$

8.5. **Intersection of CM cycles.**

Definition 8.5.1. Let $M \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 2. Define the arithmetic intersection numbers

$$\mathrm{Int}^{\mathrm{CM}}(M) := \chi\left(\widetilde{\mathcal{N}}_0(x_0), {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{CM}}(M)\right).$$

Here χ denotes the Euler–Poincaré characteristic.

Theorem 8.5.2. *Let $M \subset \mathbb{B}$ be a $\mathbb{Z}_p$-lattice of rank 2. Then*

$$\mathrm{Int}^{\mathrm{CM}}(M) = \partial\mathrm{Den}(S, M).$$

Proof. Let $L = M \obot \mathbb{Z}_p \cdot 1$. By the proof of Lemma 8.4.1, we have

$$^{\mathbb{L}}\mathcal{Z}(L) = [\mathcal{O}(0,-1) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathrm{Exc}_{\mathcal{M}}}} \mathcal{O}(0,-1)] + (\iota_0)_* {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{CM}}(M).$$

Therefore

$$\mathrm{Int}^{\mathrm{CM}}(M) = \mathrm{Int}^{\mathcal{Z}}(L) = \partial\mathrm{Den}(H_0(p), L) \overset{\text{Lemma 8.3.1}}{=} \partial\mathrm{Den}(S, M).$$

□

Proof of Conjecture 8.1.2. The original Conjecture 8.1.2 follows by combining Theorem 8.5.2 and Corollary 8.2.5. □

9. Arithmetic intersection of Hecke correspondences

9.1. **Modular curves.** Let $\mathbb{A}_f$ be the ring of finite adèles over $\mathbb{Q}$. Let $U \subset \mathrm{GL}_2(\mathbb{A}_f)$ be a sufficiently small compact open subgroup, the quotient space $\mathrm{GL}_2(\mathbb{Q})\backslash\mathbb{H} \times \mathrm{GL}_2(\mathbb{A}_f)/U$ admits a canonical model Y_U over $\mathbb{Q}$ such that

$$Y_U(\mathbb{C}) := Y_U \times_{\mathbb{Q}} \mathbb{C} = \mathrm{GL}_2(\mathbb{Q})\backslash\mathbb{H} \times \mathrm{GL}_2(\mathbb{A}_f)/U \simeq \mathrm{GL}_2(\mathbb{Q})^+\backslash\mathbb{H}^+ \times \mathrm{GL}_2(\mathbb{A}_f)/U,$$

where $\mathrm{GL}_2(\mathbb{Q})^+$ consists of elements with positive determinant and $\mathbb{H}^+$ is the upper half plane. The curve Y_U has a canonical compactification X_U which is a proper smooth curve over $\mathbb{Q}$. The curve X_U has a Hodge class $\mathscr{L}_U \in \mathrm{Pic}(X_U)_{\mathbb{Q}}$ as defined in [YZZ13, §3.1.3]

9.2. **Cyclic isogeny.** Let $\pi : E \to E'$ is an isogeny of two elliptic curves over a base scheme S. For a prime $p|\deg(\pi)$, define the p-part $\pi_p : E \to E_1$ of π to be an isogeny which factors π in the way $\pi = \pi^p \circ \pi_p$ where $\pi^p : E_1 \to E'$ is another isogeny such that $p \nmid \deg(\pi^p)$.

Definition 9.2.1. An isogeny $\pi : E \to E'$ between two elliptic curves over S is cyclic if $\ker(\pi)$ is a cyclic group scheme over S in the sense of [KM85, §6.1]. For a prime number $p|\deg(\pi)$, we say π is p-cyclic if the p-power torsion subgroup of $\ker(\pi)$ is cyclic in the sense of [KM85, §6.1].

For a cyclic isogeny $\pi : E \to E'$ of degree p^n for some integer $n \geqslant 0$ between elliptic curves. There is a standard decomposition of π into the composition of n degree p isogenies:

$$\pi : E = E_0 \xrightarrow{\pi_1} E_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_n} E_n \simeq E'.$$

We refer to [KM85, §6.7] for the details on the notions of standard decomposition.

9.3. **Global Hecke correspondences.** Let U be the same compact open subgroup in $\mathrm{GL}_2(\mathbb{A}_f)$ as §9.1. For a double coset UxU of $U\backslash\mathrm{GL}_2(\mathbb{A}_f)/U$ where $x\in\mathrm{GL}_2(\mathbb{A}_f)$, we have a Hecke correspondence defined as the image of the morphism:

$$(\pi_{U\cap xUx^{-1},U},\pi_{U\cap xUx^{-1},U}\circ T_x):\mathsf{T}(x)_U:=X_{U\cap xUx^{-1}}\to X_U^2.$$

Notice that the connected components $\pi_0(X_U)$ of X_U is given by $\mathbb{Q}_+^\times\backslash\mathbb{A}_f^\times/\det(U)$. For an element $a\in\pi_0(X_U)$, define $X_{U,a}$ to be the connected component of X_U corresponding to a. For an element $\alpha\in\mathbb{Q}_+^\times\backslash\mathbb{A}_f^\times/\det(U)$, define $M_{U,\alpha}=\bigsqcup\limits_{a\in\pi_0(X_U)}X_{U,a}\times X_{U,a\alpha}$. Then

$$X_U^2=\bigsqcup_{\alpha\in\mathbb{Q}_+^\times\backslash\mathbb{A}_f^\times/\det(U)}M_{U,\alpha}.$$

The Hecke correspondence $\mathsf{T}(x)_U$ is supported on the component $M_U:=M_{U,\det(x)}$.

9.4. **Integral models of modular curves.** Let p be a finite prime. Let n be a positive integer. Let $U^p\subset\mathrm{GL}_2(\mathbb{A}_f^p)$ is a sufficiently small compact open subgroup. Let $\Gamma_0(p^n)$ be the following subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$:

$$\Gamma_0(p^n)=\left\{g=\begin{pmatrix}a&b\\p^nc&d\end{pmatrix}:\ a,b,c,d\in\mathbb{Z}_p,\ \det(g)\in\mathbb{Z}_p^\times.\right\}$$

Let $U=\Gamma_0(p^n)\cdot U^p$, we are going to construct an integral model $\mathcal{Y}_U$ for the modular curve Y_U. The model $\mathcal{Y}_{U^p}(p^n)$ is a Deligne–Mumford stack over $\mathbb{Z}_{(p)}$. Let S be a connected $\mathbb{Z}_{(p)}$-scheme. Let $\overline{s}$ be a geometric point of S. The groupoid $\mathcal{Y}_{U^p}(p^n)(S)$ consists of objects $(E\xrightarrow{\pi}E',\overline{\eta^p})$, where $E\xrightarrow{\pi}E'$ is a cyclic isogeny of degree p^n and $\overline{\eta^p}$ is a $\pi_1(S,\overline{s})$-invariant U^p-equivalence class of isomorphisms

$$\eta^p:\left(\mathbb{A}_f^p\right)^2\xrightarrow{\simeq}V^p(E_{\overline{s}}).$$

A morphism from $(E_1\xrightarrow{\pi_1}E_1',\overline{\eta_1^p})$ to $(E_2\xrightarrow{\pi_2}E_2',\overline{\eta_2^p})$ is a pair (f,f') of isomorphisms $f:E_1\to E_2$ and $f':E_1'\to E_2'$ such that $f'\circ\pi_1=\pi_2\circ f$ and $\overline{\eta_1^p}=\overline{V^p(f)\circ\eta_2^p}$ as U^p-orbits.

For a point $\tau\in\mathbb{H}^+$, denote by E_τ the elliptic curve $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$. For the elliptic curve E_τ, we have $E[m]=\left(\frac{1}{m}\mathbb{Z}+\frac{1}{m}\mathbb{Z}\tau\right)/\mathbb{Z}+\mathbb{Z}\tau\simeq(\mathbb{Z}/m\mathbb{Z})^2$, there is naturally defined $a_\tau^p:V^p(E_\tau)\to\left(\mathbb{A}_f^p\right)^2$ by choosing the isomorphism $E[m]\xrightarrow{\sim}(\mathbb{Z}/m\mathbb{Z})^2$ compatibly using the basis $1,\tau$. There is a bijection between the set associated to the groupoid $\mathcal{Y}_U(\mathbb{C})$ and $Y_U(\mathbb{C})$ given an follows: Let $[\tau,g]\in Y_U(\mathbb{C})\simeq\mathrm{GL}_2(\mathbb{Q})^+\backslash\mathbb{H}^+\times\mathrm{GL}_2(\mathbb{A}_f)/U$ be a point. It is mapped to the object

$$\left(E_\tau\xrightarrow{\times p^n}E_{p^n\tau},\overline{(a_\tau^p)^{-1}\circ g}\right)\in\mathcal{Y}_U(\mathbb{C}).\tag{85}$$

Given an object $(E\xrightarrow{\pi}E',\overline{\eta^p})\in\mathcal{Y}_U(\mathbb{C})$, there exists an element $\tau\in\mathbb{H}^+$ such that there exists two isomorphisms $f:E\to E_\tau$ and $f':E'\to E_{p^n\tau}$ such that $f'\circ\pi\circ f^{-1}:E_\tau\to E_{p^n\tau}$ is given by multiplication by p^n. This object is mapped to the following element in $Y_U(\mathbb{C})$:

$$[\tau,a_\tau^p\circ V^p(f)\circ\eta^p]\in\mathrm{GL}_2(\mathbb{Q})^+\backslash\mathbb{H}^+\times\mathrm{GL}_2(\mathbb{A}_f)/U.\tag{86}$$

The above two maps are inverse to each other.

The stack $\mathcal{Y}_U$ is a 2-dimensional Deligne–Mumford stack. There exists another 2-dimensional regular Deligne–Mumford stack $\mathcal{X}_U$ which can be viewed as a compactification of $\mathcal{Y}_U$. In particular, $\mathcal{X}_U$ is proper over $\mathbb{Z}_{(p)}$. We refer the readers to [Č17, §4.1.2] for more details about $\mathcal{X}_U$. The stack $\mathcal{Y}_U$ is an open substack of $\mathcal{X}_U$ and $\mathcal{X}_U(\mathbb{C}) \simeq X_U(\mathbb{C})$.

9.5. **Moduli interpretations of Hecke correspondences.**

Lemma 9.5.1. *The following elements form a set of coset representative for* $\Gamma_0(p)\backslash \mathrm{GL}_2(\mathbb{Q}_p)/\Gamma_0(p)$*:*

$$\begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}, \quad \begin{pmatrix} 0 & p^a \\ p^b & 0 \end{pmatrix},$$

where $a, b \in \mathbb{Z}$.

Proof. We refer to the proof of Bushnell and Henniart [BH06, §17.1]. □

Definition 9.5.2. We divide the group $\mathrm{GL}_2(\mathbb{Q}_p)$ into the following four subsets

$$\mathbb{V}_p^{\mathrm{I}+} = \bigcup_{a \leqslant b} \Gamma_0(p) \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix} \Gamma_0(p), \quad \mathbb{V}_p^{\mathrm{I}-} = \bigcup_{b < a} \Gamma_0(p) \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix} \Gamma_0(p),$$
$$\mathbb{V}_p^{\mathrm{II}+} = \bigcup_{a < b} \Gamma_0(p) \begin{pmatrix} 0 & p^a \\ p^b & 0 \end{pmatrix} \Gamma_0(p), \quad \mathbb{V}_p^{\mathrm{II}-} = \bigcup_{b \leqslant a} \Gamma_0(p) \begin{pmatrix} 0 & p^a \\ p^b & 0 \end{pmatrix} \Gamma_0(p).$$

Lemma 9.5.3. *Let* $x \in \mathrm{M}_2(\mathbb{A}_f)$ *such that* $\det(x) \in \mathbb{Q}_+^\times$ *and* x_p *is a primitive element in the lattice* $H_0(p)$*. Let* $x^* \in \mathrm{M}_2(\mathbb{A}_f)$ *be the contragredient of* x*. Let* $n = \nu_p(\det(x_p)) \geqslant 0$*. Let* $U = \Gamma_0(p) \cdot U^p$ *be a compact open subgroup of* $\mathrm{GL}_2(\mathbb{A}_f)$*. Then*

- I+: *If* $x_p \in \mathbb{V}_p^{\mathrm{I}+}$*, there exists an isomorphism* $\mathsf{T}(x)_U \simeq \mathcal{X}_{\Gamma_0(p^{n+1}) \cdot U^p \cap x U^p x^{-1}}$ *under which the morphism* $\mathsf{T}(x)_U \to X_U^2$ *on the open subvariety* $Y_{\Gamma_0(p^{n+1}) \cdot U^p \cap x U^p x^{-1}}$ *is given by*

$$\iota_x^{\mathrm{I}+} : \left(E \xrightarrow{\pi} E', \overline{\eta^p}\right) \mapsto \left(\left(E \xrightarrow{\pi_1} E_1, \overline{\eta^p}\right), \left(E_n \xrightarrow{\pi_{n+1}} E', \overline{V^p(\pi_n \circ \cdots \circ \pi_1) \circ \eta^p \circ (x^*)^{-1}}\right)\right).$$

- I−: *If* $x_p \in \mathbb{V}_p^{\mathrm{I}-}$*, there exists an isomorphism* $\mathsf{T}(x)_U \simeq \mathcal{X}_{\Gamma_0(p^{n+1}) \cdot U^p \cap x^{-1} U^p x}$ *under which the morphism* $\mathsf{T}(x)_U \to X_U^2$ *on the open subvariety* $Y_{\Gamma_0(p^{n+1}) \cdot U^p \cap x^{-1} U^p x}$ *is given by*

$$\iota_x^{\mathrm{I}-} : \left(E \xrightarrow{\pi} E', \overline{\eta^p}\right) \mapsto \left(\left(E_n \xrightarrow{\pi_{n+1}} E', \overline{V^p(\pi_n \circ \cdots \circ \pi_1) \circ \eta^p \circ x^{-1}}\right), \left(E \xrightarrow{\pi_1} E_1, \overline{\eta^p}\right)\right).$$

- II+: *If* $x_p \in \mathbb{V}_p^{\mathrm{II}+}$*, there exists an isomorphism* $\mathsf{T}(x)_U \simeq \mathcal{X}_{\Gamma_0(p^n) \cdot U^p \cap x U^p x^{-1}}$ *under which the morphism* $\mathsf{T}(x)_U \to X_U^2$ *on the open subvariety* $Y_{\Gamma_0(p^n) \cdot U^p \cap x U^p x^{-1}}$ *is given by*

$$\iota_x^{\mathrm{II}+} : \left(E \xrightarrow{\pi} E', \overline{\eta^p}\right) \mapsto \left(\left(E \xrightarrow{\pi_1} E_1, \overline{\eta^p}\right), \left(E' \xrightarrow{\pi_n^\vee} E_{n-1}, \overline{V^p(\pi) \circ \eta^p \circ (x^*)^{-1}}\right)\right).$$

- II−: *If* $x_p \in \mathbb{V}_p^{\mathrm{II}-}$*, there exists an isomorphism* $\mathsf{T}(x)_U \simeq \mathcal{X}_{\Gamma_0(p^n) \cdot U^p \cap x^{-1} U^p x}$ *under which the morphism* $\mathsf{T}(x)_U \to X_U^2$ *on the open subvariety* $Y_{\Gamma_0(p^n) \cdot U^p \cap x^{-1} U^p x}$ *is given by*

$$\iota_x^{\mathrm{II}-} : \left(E \xrightarrow{\pi} E', \overline{\eta^p}\right) \mapsto \left(\left(E_1 \xrightarrow{\pi_1^\vee} E, \overline{V^p(\pi_1) \circ \eta^p \circ x^{-1}}\right), \left(E_{n-1} \xrightarrow{\pi_n} E', \overline{V^p(\pi_{n-1} \circ \cdots \circ \pi_1) \circ \eta^p}\right)\right).$$

Proof. We only give the proof for the case I+. For simplicity, denote by x_n the following element

$$x_n = \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})^+.$$

We have the following diagram for the correspondence $\mathsf{T}(x)_U$:

$$\begin{array}{ccc} & \mathsf{T}(x)_U & \\ \swarrow^{\pi} & & \searrow^{\pi\circ T(x)} \\ X_{\Gamma_0(p)\cdot U^p} & & X_{\Gamma_0(p)\cdot U^p} \end{array}$$

By the definition of Hecke correspondence $\mathsf{T}(x)_U$, we know that it is isomorphic to compactification of the modular curve $Y_{\Gamma_0(p^{n+1})\cdot U^p \cap xU^px^{-1}}(\mathbb{C}) \simeq \mathrm{GL}_2(\mathbb{Q})^+\backslash \mathbb{H}^+ \times \mathrm{GL}_2(\mathbb{A}_f)/U^p \cap xU^px^{-1}$. Let $[\tau, g]$ be an element of the set $Y_{\Gamma_0(p^{n+1})\cdot U^p \cap xU^px^{-1}}(\mathbb{C})$, it is mapped to $[\tau, g] \in X_{\Gamma_0(p)\cdot U^p}(\mathbb{C})$ under the morphism π, and $[\tau, gx] = [x_n\tau, x_n gx] = [\det(x)^{-1}x_n z, x_n g\,(x^*)^{-1}] \in X_{\Gamma_0(p)\cdot U^p}(\mathbb{C})$ under the morphism $\pi \circ T(x)$.

By the bijective map (85), the element $[\tau, g] \in Y_{\Gamma_0(p^{n+1})\cdot U^p \cap xU^px^{-1}}(\mathbb{C})$ corresponds to the object $\left(E_\tau \xrightarrow{\times p^{n+1}} E_{p^{n+1}\tau}, \overline{(a_\tau^p)^{-1} \circ g}\right) \in \mathcal{Y}_{\Gamma_0(p^{n+1})\cdot U^p \cap xU^px^{-1}}(\mathbb{C})$ in the groupoid $\mathcal{Y}_{\Gamma_0(p^{n+1})\cdot U^p \cap xU^px^{-1}}(\mathbb{C})$. The element $[\tau, g] \in Y_{\Gamma_0(p)\cdot U^p}(\mathbb{C})$ corresponds to the object $\left(E_\tau \xrightarrow{\times p} E_{p\tau}, \overline{(a_\tau^p)^{-1} \circ g}\right) \in \mathcal{Y}_{\Gamma_0(p)\cdot U^p}(\mathbb{C})$ in the groupoid $\mathcal{Y}_{\Gamma_0(p)\cdot U^p}(\mathbb{C})$. The element $[\det(x)^{-1}x_n\tau, x_n g\,(x^*)^{-1}] \in Y_{\Gamma_0(p)\cdot U^p}(\mathbb{C})$ corresponds to the object $\left(E_{p^n\tau} \xrightarrow{\times p} E_{p^{n+1}\tau}, \overline{(a_{p^n\tau}^p)^{-1} \circ x_n g\,(x^*)^{-1}}\right) \in \mathcal{Y}_{\Gamma_0(p)\cdot U^p}(\mathbb{C})$ in the groupoid $\mathcal{Y}_{\Gamma_0(p)\cdot U^p}(\mathbb{C})$. Notice that $E_\tau \xrightarrow{\times p^n} E_{p^n\tau}$ is the composition of the first n degree p isogenies of the standard decomposition of the degree p^{n+1} cyclic isogeny $E_\tau \xrightarrow{\times p^{n+1}} E_{p^{n+1}\tau}$. We have the following commutative diagram

$$\begin{array}{ccccc} \left(\mathbb{A}_f^p\right)^2 & \xrightarrow{\quad \eta^p \quad} & V^p(E_\tau) & \xrightarrow{a_\tau^p} & \left(\mathbb{A}_f^p\right)^2 \\ \downarrow{\scriptstyle x^*} & & \downarrow{\scriptstyle V^p(\times p^n)} & & \downarrow{\scriptstyle x_n} \\ \left(\mathbb{A}_f^p\right)^2 & \xrightarrow{V^p(\times p^n)\circ\eta^p\circ(x^*)^{-1}} & V^p(E_{p^n\tau}) & \xrightarrow{a_{p^n\tau}^p} & \left(\mathbb{A}_f^p\right)^2 . \end{array}$$

Therefore $\left(a_{p^n\tau}^p\right)^{-1} \circ x_n g\,(x^*)^{-1} = V^p(\times p^n) \circ \eta^p \circ (x^*)^{-1}$. Hence the object $\left(E \xrightarrow{\pi} E', \overline{\eta^p}\right) \in \mathcal{Y}_{\Gamma_0(p^{n+1})\cdot U^p \cap xU^px^{-1}}(\mathbb{C})$ is mapped to $\left(E \xrightarrow{\pi_1} E_1, \overline{\eta^p}\right), \left(E_n \xrightarrow{\pi_{n+1}} E', \overline{V^p(\pi_n \circ \cdots \circ \pi_1) \circ \eta^p \circ (x^*)^{-1}}\right) \in \mathcal{Y}_{\Gamma_0(p)\cdot U^p}(\mathbb{C})$, i.e., the case I+ is true. □

9.6. **Generating series.** Let v be a place of $\mathbb{Q}$. Let $\mathbb{A}$ be the ring of adèles over $\mathbb{Q}$. Let $V = \mathrm{M}_2(\mathbb{Q})$ be equipped with the quadratic form given by determinant. Recall that $\mathbb{V} = \{\mathbb{V}_v\}$ is the following incoherent collection of quadratic spaces of $\mathbb{A}$ of rank 4 (cf. (7)),

$$\mathbb{V}_v = V_v = \mathrm{M}_2(\mathbb{Q}_v) \text{ if } v < \infty, \text{ and } \mathbb{V}_\infty \text{ is positive definite.} \tag{87}$$

Let $q_{\mathbb{V}} : \mathbb{V} \to \mathbb{A}$ be the quadratic form on $\mathbb{V}$. Let $\mathbb{V}_f = \mathbb{V} \otimes_{\mathbb{A}} \mathbb{A}_f$. Let $H = \mathrm{GSpin}(V)$, we have $H \simeq \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$. We still use det to denote the morphism $H \to \mathbb{G}_m$.

Let $U \subset \mathrm{GL}_2(\mathbb{A}_f)$ be a compact open subgroup. Let $K = (U \times U) \cap H(\mathbb{A}_f) \simeq U \times_{\mathbb{G}_m} U$. Let M_K be the compactification of the Shimura variety $\mathrm{Sh}(H, K)$. It's easy to see that $M_K(\mathbb{C}) = M_{U,1}$. For $i = 1, 2$, let $p_i : M_K \to X_U$ be the two projection morphisms. Let $x = x_\infty \otimes x_f \in \mathbb{V}$ be an element.

Following [YZZ23, §3.3], define the following cycle in M_K:

$$\mathsf{T}(x)_K = \begin{cases} \mathsf{T}(x_f)_U, & \text{if } q_{\mathbb{V}}(x) \in \mathbb{Q}^\times; \\ \frac{1}{2}\left(p_1^*\mathscr{L}_U + p_2^*\mathscr{L}_U\right), & \text{if } x = 0; \\ 0, & \text{if } q_{\mathbb{V}}(x) \notin \mathbb{Q}^\times \text{ and } x \neq 0. \end{cases}$$

Let $\mathscr{S}(\mathbb{V}) = \mathscr{S}(\mathbb{V}_\infty) \otimes \mathscr{S}(\mathbb{V}_f)$ be the Schwartz function space on $\mathbb{V}$. Let $\phi_\infty \in \mathscr{S}(\mathbb{V}_\infty)$ be the standard Gaussian function. Let $\phi_f \in \mathscr{S}(\mathbb{V}_f)$ be a K-invariant function. Let $\phi = \phi_\infty \otimes \phi_f$. Let $\widetilde{K} = \mathrm{O}(\mathbb{V}_\infty) \times K$, it acts on $\mathbb{V}$. Define the following generating series

$$\mathsf{T}(\phi) = \sum_{x \in \widetilde{K} \backslash \mathbb{V}} \phi(x) \mathsf{T}(x)_K. \tag{88}$$

Notice that there exists an extended Weil representation r of $\mathrm{GL}_2(\mathbb{A})$ on the space $\mathscr{S}(\mathbb{V})$ (cf. [YZZ23, §2.1]). Define

$$\mathsf{T}(g, \phi) = \sum_{x \in \widetilde{K} \backslash \mathbb{V}} \left(r(g)\phi\right)(x) \cdot \mathsf{T}(x)_K. \tag{89}$$

9.7. **A regular integral model for M_K and supersingular uniformization.** Let $U = \Gamma_0(p) \cdot U^p$, where $U^p \subset \mathrm{GL}_2(\mathbb{A}_f^p)$ is a sufficiently small compact open subgroup. Let $K = U \times_{\mathbb{G}_m} U$ and $K^p = U^p \times_{\mathbb{G}_m} U^p$, we will construct a regular integral model $\mathcal{M}_{K,(p)}$ for the compactified Shimura variety M_K over $\mathbb{Z}_{(p)}$.

Let $\mathcal{H}^\circ_{K,(p)}$ be the integral model of the Shimura variety associated to the group $H = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ of level K. For a connected $\mathbb{Z}_{(p)}$-scheme S and a geometric point $\overline{s}$ of S, $\mathcal{H}^\circ_{K,(p)}(S)$ is a groupoid whose objects are of the following form

$$\left(\left(E_1 \xrightarrow{\pi_1} E_1'\right), \left(E_2 \xrightarrow{\pi_2} E_2'\right), \overline{(\eta_1^p, \eta_2^p)}\right) \tag{90}$$

where π_1, π_2 are degree p isogenies between elliptic curves, and $\overline{(\eta_1^p, \eta_2^p)}$ is a $\pi_1(S, \overline{s})$-invariant K^p-orbit of a pair of isomorphisms:

$$\eta_i^p : T^p(E_i) := \prod_{l \neq p} T_l(E_i) \xrightarrow{\sim} \left(\widehat{\mathbb{Z}}^p\right)^2, \ i = 1, 2,$$

such that for the elements $e_1 = (1, 0)$ and $e_2 = (0, 1)$ of $\left(\widehat{\mathbb{Z}}^p\right)^2$, we have

$$\left((\eta_1^p)^{-1}(e_1), (\eta_1^p)^{-1}(e_2)\right)_{E_1} = \left((\eta_2^p)^{-1}(e_1), (\eta_2^p)^{-1}(e_2)\right)_{E_2},$$

where $(\cdot, \cdot)_{E_i}$ is the Weil pairing on the elliptic curve E_i for $i = 1, 2$.

Denote by $\mathcal{H}_{K,(p)}$ the compactification of the integral model $\mathcal{H}^\circ_{K,(p)}$ in the sense of [Lan13]. It can be viewed as an integral model of M_K. Denote by $\mathcal{H}^{\mathrm{ss}}_{K,(p)} \subset \mathcal{H}^\circ_{K,(p)} \subset \mathcal{H}_{K,(p)}$ the closed substack where E_1 and E_2 are supersingular. Let $\mathbb{E}$ be a supersingular elliptic curve over $\mathbb{F}$ such that $\mathbb{E}[p^\infty] \simeq \mathbb{X}$. Let B be the unique quaternion division algebra over $\mathbb{Q}$ which ramifies at p and ∞ with norm map $\nu_B : B \to \mathbb{Q}$. Let

$$\left(\left(\mathbb{E}_1 \xrightarrow{f_1} \mathbb{E}_1'\right), \left(\mathbb{E}_2 \xrightarrow{f_2} \mathbb{E}_2'\right), \overline{(\eta_{\mathbb{E},1}^p, \eta_{\mathbb{E},2}^p)}\right)$$

be one of the objects in $\mathcal{H}_{K,(p)}^{\mathrm{ss}}(\mathbb{F})$. Then there exists four quasi-isogenies $\rho_{\mathbb{E},i} : \mathbb{E} \to \mathbb{E}_i$, $\rho'_{\mathbb{E},i} : \mathbb{E} \to \mathbb{E}'_i$ $(i = 1, 2)$ of degrees 1 such that $\rho'^{-1}_{\mathbb{E},i} \circ f_i \circ \rho_{\mathbb{E},i} = x_0$. We fix a choice of representatives $\eta^p_{\mathbb{E},1}$ and $\eta^p_{\mathbb{E},2}$ from now on.

Let $\widehat{\mathcal{H}}_K^{\mathrm{ss}}$ be the base change to $\breve{\mathbb{Z}}_p$ of the completion of $\mathcal{H}_{K,(p)}$ along the closed substack $\mathcal{H}_{K,(p)}^{\mathrm{ss}}$ (we refer to [Har05, Definition A.12] for the definition of the completion of an algebraic stack along a closed substack, see also [Eme16, Example 5.9, Remark 5.10]). Then we have the following supersingular uniformization map which is an isomorphism between formal algebraic stacks over $\breve{\mathbb{Z}}_p$ ([RZ96, Theorem 6.24]):

$$\Theta_{\mathcal{H}} : \widehat{\mathcal{H}}_K^{\mathrm{ss}} \xrightarrow{\sim} H'(\mathbb{Q})_0 \backslash \mathcal{N}(x_0) \times H(\mathbb{A}_f^p)/K^p, \tag{91}$$

where $H'(\mathbb{Q})_0 = \{g \in B^\times(\mathbb{Q}) \times_{\mathbb{Q}^\times} B^\times(\mathbb{Q}) : \nu_p(\nu_B(g)) = 0\}$. The isomorphism should be viewed as an isomorphism between fppf sheaves of groupoids over $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$. The map $\Theta_{\mathcal{H}}$ is given as follows: Let S be an object in Nilp_W. Let

$$\left(\left(E_1 \xrightarrow{\pi_1} E'_1\right), \left(E_2 \xrightarrow{\pi_2} E'_2\right), \overline{(\eta_1^p, \eta_2^p)}\right) \tag{92}$$

be an object in $\widehat{\mathcal{H}}_K^{\mathrm{ss}}(S)$. Then there exists four quasi-isogenies $\rho_{E,i} : \mathbb{E}_i \times_{\mathbb{F}} \overline{S} \to E_i \times_S \overline{S}$, $\rho'_{E,i} : \mathbb{E}'_i \times_{\mathbb{F}} \overline{S} \to E'_i \times_S \overline{S}$ $(i = 1, 2)$ of degrees 1 such that $\pi_i \circ \rho_{E,i} = \rho'_{E,i} \circ f_i$ for $i = 1, 2$. For $i = 1, 2$, let $g_i \in \mathrm{GL}_2(\mathbb{A}_f^p)$ be the element such that

$$g_i = \left(\eta^p_{\mathbb{E},i}\right)^{-1} \circ V^p(\rho_{E,i})^{-1} \circ \eta_i^p. \tag{93}$$

Then g_i is determined up to right multiplication by an element in U^p. By the definition of $\mathcal{H}_{K,(p)}$, we must have $\det(g_1) = \det(g_2)$. Let $\rho_i = (\rho_{E,i} \circ \rho_{\mathbb{E},i})[p^\infty]$ and $\rho'_i = \left(\rho^\infty_{E,i} \circ \rho^\infty_{\mathbb{E},i}\right)[p^\infty]$. Then the object (92) is mapped to

$$\begin{aligned}\left(\left(E_1[p^\infty] \xrightarrow{\pi_1[p^\infty]} E'_1[p^\infty], (\rho_1, \rho'_1)\right), \left(E_2[p^\infty] \xrightarrow{\pi_2[p^\infty]} E'_2[p^\infty], (\rho_2, \rho'_2)\right)\right) \times (g_1, g_2) \\ \in H'(\mathbb{Q})_0 \backslash \mathcal{N}(x_0)(S) \times H(\mathbb{A}_f^p)/K^p.\end{aligned}$$

We also want to make the action of $H'(\mathbb{Q})_0$ on the formal scheme $\mathcal{N}(x_0)$ clear. Let

$$\left(X_1 \xrightarrow{\pi_1} X'_1, (\rho_1, \rho'_1)\right), \left(X_2 \xrightarrow{\pi_2} X'_2, (\rho_2, \rho'_2)\right) \tag{94}$$

be an object in the set $\mathcal{N}(x_0)(S)$. Let $(b_1, b_2) \in H'(\mathbb{Q})_0$. Then

$$\begin{aligned}(b_1, b_2)\cdot &\left(\left(X_1 \xrightarrow{\pi_1} X'_1, (\rho_1, \rho'_1)\right), \left(X_2 \xrightarrow{\pi_2} X'_2, (\rho_2, \rho'_2)\right)\right) \\ &= \left(\left(X_1 \xrightarrow{\pi_1} X'_1, (\rho_1 b_1^{-1}, \rho'_1 b'^{-1}_1)\right), \left(X_2 \xrightarrow{\pi_2} X'_2, (\rho_2 b_2^{-1}, \rho'_2 b'^{-1}_2)\right)\right).\end{aligned}$$

Let $\mathcal{M}_{K,(p)}$ be the blow up of the Deligne–Mumford stack $\mathcal{H}_{K,(p)}$ along the closed substack $\mathcal{H}_{K,(p)}^{\mathrm{ss}}$, i.e., $\mathcal{M}_{K,(p)}$ is the stacky proj of the Rees algebra $\bigoplus_{n \geqslant 0} \mathcal{I}^n$ over $\mathcal{H}_{K,(p)}$ where $\mathcal{I}$ is the ideal sheaf of the closed substack $\mathcal{H}_{K,(p)}^{\mathrm{ss}}$ ([QR22, Definition 3.2.1, Example 3.2.6]). Let $\widehat{\mathcal{M}}_K^{\mathrm{ss}}$ be the base change to $\breve{\mathbb{Z}}_p$ of the completion of $\mathcal{M}_{K,(p)}$ along the exceptional divisor Exc_K. Notice that by the universal property of blow up, the automorphism $g : \mathcal{N}(x_0) \to \mathcal{N}(x_0)$ extends to an automorphism $g : \mathcal{M} \to \mathcal{M}$, i.e., the group $H'(\mathbb{Q})_0$ also acts on the formal scheme $\mathcal{M}$.

Notice that if the level group $K^p \subset H(\mathbb{A}_f^p)$ is sufficiently small, the Deligne–Mumford stacks $\mathcal{H}_{K,(p)}$ and $\mathcal{H}_{K,(p)}^{\mathrm{ss}}$ are schemes, the morphism $\pi : \mathcal{M}_{K,(p)} \to \mathcal{H}_{K,(p)}$ is the usual blow up morphism along the closed subscheme $\mathcal{H}_{K,(p)}^{\mathrm{ss}}$. The formal algebraic stacks $\widehat{\mathcal{M}}_K^{\mathrm{ss}}$ and $\widehat{\mathcal{H}}_K^{\mathrm{ss}}$ are obtained by formal completion along Exc_K and $\mathcal{H}_{K,(p)}^{\mathrm{ss}}$ respectively.

Lemma 9.7.1. *The stack $\mathcal{M}_{K,(p)}$ is a 3-dimensional regular Deligne–Mumford stack. There exists a supersingular uniformization map which is an isomorphism between formal algebraic stacks over $\breve{\mathbb{Z}}_p$ (or simply an isomorphism between two fppf sheaves of groupoids over* $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$*):*

$$\Theta_{\mathcal{M}} : \widehat{\mathcal{M}}_K^{\mathrm{ss}} \xrightarrow{\sim} H'(\mathbb{Q})_0\backslash\mathcal{M} \times H(\mathbb{A}_f^p)/K^p. \tag{95}$$

It also makes the following diagram commutative

$$\begin{array}{ccc} \widehat{\mathcal{M}}_K^{\mathrm{ss}} & \xrightarrow{\Theta_{\mathcal{M}}} & H'(\mathbb{Q})_0\backslash\mathcal{M} \times H(\mathbb{A}_f^p)/K^p \\ \downarrow & & \downarrow \\ \widehat{\mathcal{H}}_K^{\mathrm{ss}} & \xrightarrow{\Theta_{\mathcal{H}}} & H'(\mathbb{Q})_0\backslash\mathcal{N}(x_0) \times H(\mathbb{A}_f^p)/K^p. \end{array}$$

Proof. Let $x \in |\mathcal{M}_{K,(p)}(\mathbb{F})|$ be a point. Let $\mathcal{O}_x$ denote the étale local ring of the Deligne–Mumford stack $\mathcal{M}_{K,(p)}$ at x.

If $x \notin |\mathrm{Exc}_K(\mathbb{F})|$, let $x' = \pi(x) \in |\mathcal{H}_{K,(p)}(\mathbb{F})|$, we have $\mathcal{O}_x \simeq \mathcal{O}_{x'}$. Let $x' = (x_1, x_2)$ where $x_i \in |\mathcal{X}_U(\mathbb{F})|$. The local ring $\mathcal{O}_{x_i}$ is the strict henselization of a regular local ring A which is flat over $\mathbb{Z}_p$. Moreover, the ring A is formally smooth over $\mathbb{Z}_p$ if and only if $x_i \notin |\mathcal{Y}_U^{\mathrm{ss}}(\mathbb{F})|$ by the works of Katz and Mazur [KM85], Edixhoven [Edi90]. We refer to [Edi90, §1.1.3] for the formally smoothness at an ordinary point, and §1.2.3 of *loc. cit.* for the formally smoothness at a cusp point (here we use the fact that the level structure of $\mathcal{X}_U$ at p is given by $\Gamma_0(p)$).

- If $x' \notin |\mathcal{H}_{K,(p)}^{\circ}(\mathbb{F})|$, we have at least one of x_i $(i = 1, 2)$ belongs to the cuspidal locus of the modular curve $\mathcal{X}_U$, hence the local ring $\mathcal{O}_{x'}$ is the strict henselization of a local ring of the form $A_1 \otimes_{\mathbb{Z}_p} A_2$ where A_1 is formally smooth over $\mathbb{Z}_p$ and A_2 is a regular local ring.
- If $x' \in |\mathcal{H}_{K,(p)}^{\circ}(\mathbb{F})|$, we have at least one of x_i $(i = 1, 2)$ belongs to the ordinary locus of the modular curve $\mathcal{X}_U$, hence the local ring $\mathcal{O}_{x'}$ is also the strict henselization of a local ring of the form $A_1 \otimes_{\mathbb{Z}_p} A_2$ where A_1 is formally smooth over $\mathbb{Z}_p$ and A_2 is regular.

Therefore the local ring $\mathcal{O}_x \simeq \mathcal{O}_{x'}$ is the strict henselization of a regular local ring, and hence is itself regular.

If $x \in |\mathrm{Exc}_K(\mathbb{F})|$, the local ring $\mathcal{O}_x$ is the strict henselization of a local ring isomorphic to some local ring of the formal scheme $\mathcal{M}$. In both cases, the local ring $\mathcal{O}_x$ is regular by Proposition 4.11.1, hence $\mathcal{M}_{K,(p)}$ is regular.

The formal scheme $\mathcal{M}$ and the integral model $\mathcal{M}_{K,(p)}$ are constructed by blowing up along the supersingular $\mathbb{F}$-point (objects) of the formal scheme $\mathcal{N}(x_0)$ and the integral model $\mathcal{H}_{K,(p)}$. Since we have the uniformization morphism $\Theta_{\mathcal{H}} : \widehat{\mathcal{H}}_K^{\mathrm{ss}} \xrightarrow{\sim} H'(\mathbb{Q})_0\backslash\mathcal{N}(x_0) \times H(\mathbb{A}_f^p)/K^p$, the supersingular uniformization morphism $\Theta_{\mathcal{M}}$ for $\widehat{\mathcal{M}}_K^{\mathrm{ss}}$ follows easily. □

9.8. **Supersingular uniformization of Hecke correspondences.** Fix an odd prime p, we still assume that $U = \Gamma_0(p) \cdot U^p$, where $U^p \subset \mathrm{GL}_2(\mathbb{A}_f^p)$ is a sufficiently small compact open subgroup. We abbreviate $\mathcal{H}_{K,(p)}$ and $\mathcal{M}_{K,(p)}$ as $\mathcal{H}_K$ and $\mathcal{M}_K$. Let $x \in \mathbb{V}$ be an element such that $q_{\mathbb{V}}(x) \in \mathbb{Q}^{\times}$ and

x_p is a primitive element in the lattice $H_0(p)$. Let $n = \nu_p(q_{\mathbb{V}}(x)) \geqslant 0$. Let symbols $* \in \{\mathrm{I}, \mathrm{II}\}$ and $? \in \{+, -\}$. If $x_p \in \mathbb{V}_p^{*?}$ is a primitive element in $H_0(p)$, we can extend the morphism $\mathsf{T}(x_f)_U \to X_U^2$ to $\mathcal{X}_{\Gamma_0(p^{n+1})\cdot U'^p} \to \mathcal{H}_K$ (if $* = \mathrm{I}$) or $\mathcal{X}_{\Gamma_0(p^n)\cdot U'^p} \to \mathcal{H}_K$ (if $* = \mathrm{II}$) where $U'^p = U^p \cap xU^px^{-1}$ (if $? = +$) or $U'^p = U^p \cap x^{-1}U^px$ (if $? = -$) using the moduli interpretations of Lemma 9.5.3. Define

$$\widehat{\mathsf{T}}(x)_K = \begin{cases} \left(\text{strict transform of } \mathcal{X}_{\Gamma_0(p^{n+1})\cdot U'^p}\right) + \mathrm{Exc}_K, & \text{if } * = \mathrm{I}; \\ \text{strict transform of } \mathcal{X}_{\Gamma_0(p^n)\cdot U'^p}, & \text{if } * = \mathrm{II}. \end{cases}$$

We remark here that the symbol $\widehat{\mathsf{T}}$ is used to denote the "integral model" of Hecke correspondence, not a completion of some stuff, to keep in consistence with the notations of [YZZ23] where they added the "hat" symbol to denote an arithmetic cycle (although we haven't added a green current to it).

Definition 9.8.1. Let $x \in \mathbb{V}$ be an element such that $q_{\mathbb{V}}(x) \in \mathbb{Q}^\times$. Then there exists a unique integer r such that $p^r x_p$ is a primitive element in the lattice $H_0(p)$. Define

$$\widehat{\mathsf{T}}(x)_K = \widehat{\mathsf{T}}(p^r x)_K.$$

For a K-invariant Schwartz function $\phi_f \in \mathscr{S}(\mathbb{V}_f)$. Let $\phi = \phi_\infty \otimes \phi_f \in \mathscr{S}(\mathbb{V})$ where $\phi_\infty \in \mathscr{S}(\mathbb{V}_\infty)$ is the standard Gaussian function, i.e., $\phi_\infty(x) = e^{-2\pi q_\infty(x)}$. Define formally

$$\widehat{\mathsf{T}}(g, \phi) = \sum_{x \in \widetilde{K}\backslash \mathbb{V}} \left(r(g)\phi\right)(x) \cdot \widehat{\mathsf{T}}(x)_K, \;\; g \in \mathrm{GL}_2(\mathbb{A}). \tag{96}$$

This can be viewed as the integral version of the generating series $\mathsf{T}(g, \phi)$ as in (89) by the following lemma. The convergence of the generating series is not a prior clear, but each of its Fourier coefficients defines a codimension 1 cycle on $\mathcal{M}_K$, as they involve only finite sums.

Lemma 9.8.2. *Let $x \in \mathbb{V}$ be an element such that $q_{\mathbb{V}}(x) \in \mathbb{Q}^\times$. Then $\widehat{\mathsf{T}}(x)_K$ is an effective Cartier divisor on the Deligne–Mumford stack $\mathcal{M}_K$. Moreover, we have*

$$\widehat{\mathsf{T}}(x)_K \times_{\operatorname{Spec} \mathbb{Z}_{(p)}} \operatorname{Spec} \mathbb{Q} = \mathsf{T}(x)_K.$$

Proof. The strict transform of the regular Deligne–Mumford stack $\mathcal{X}_{\Gamma_0(p^n)\cdot U'^p}$ is still regular by Lemma 4.5.1, hence it's still a divisor on $\mathcal{M}_{K,(p)}$. Therefore $\widehat{\mathsf{T}}(x)_K$ is still a divisor.

The Hecke correspondence only depends on the coset $\mathbb{Q}^\times\backslash\mathbb{V}$. Let r be an integer such that $p^r x_p$ is a primitive element in $H_0(p)$. Then we have $\mathsf{T}(x)_K = \mathsf{T}(p^r x)_K$. Since $\mathrm{Exc}_{\mathcal{M}_K}$ is supported on the special fiber of $\mathcal{M}_{K,(p)}$, we have

$$\widehat{\mathsf{T}}(x)_K \times_{\operatorname{Spec} \mathbb{Z}_{(p)}} \operatorname{Spec} \mathbb{Q} = \widehat{\mathsf{T}}(p^r x)_K \times_{\operatorname{Spec} \mathbb{Z}_{(p)}} \operatorname{Spec} \mathbb{Q} = \mathsf{T}(p^r x)_K = \mathsf{T}(x)_K.$$

□

Recall that B is the unique quaternion division algebra over $\mathbb{Q}$ which ramifies at p and ∞. For an element $y \in B$, let $H'_y \subset H'$ be the subgroup which stabilizes y. Let $H'_y(\mathbb{Q})_0 = H'_y \cap H'(\mathbb{Q})_0$.

Lemma 9.8.3. *Let $x \in \mathbb{V}$ be an element such that $q_{\mathbb{V}}(x) \in \mathbb{Q}^\times$ and x_p is a primitive element in the lattice $H_0(p)$. Let $\widehat{\mathsf{T}}(x)_K^{\mathrm{ss}}$ be the base change to $\breve{\mathbb{Z}}_p$ of the completion of $\widehat{\mathsf{T}}(x)_K$ along its supersingular* *locus. Then we have the following identity in* $K_0^{\widehat{\mathsf{T}}(x)^{\mathrm{ss}}_K}(\widehat{\mathcal{M}}^{\mathrm{ss}}_K)_{\mathbb{C}}$*:*

$$\widehat{\mathsf{T}}(x)^{\mathrm{ss}}_K = \sum_{\substack{y\in H'(\mathbb{Q})_0\backslash B\\ q_B(y)=q_{\mathbb{V}}(x)}} \sum_{\substack{g\in H'_y(\mathbb{Q})_0\backslash H(\mathbb{A}_f^p)/K^p\\ g^{-1}y\in K^px}} \Theta_{\mathcal{M}}^{-1}\begin{cases}\left(\mathcal{N}_0^{\mathrm{I}+}(x_0y)+\mathrm{Exc}_{\mathcal{M}},g\right), & \text{if } x_p\in\mathbb{V}_p^{\mathrm{I}+};\\ \left(\mathcal{N}_0^{\mathrm{I}-}(x_0\overline{y})+\mathrm{Exc}_{\mathcal{M}},g\right), & \text{if } x_p\in\mathbb{V}_p^{\mathrm{I}-};\\ \left(\mathcal{N}_0^{\mathrm{II}+}(y),g\right), & \text{if } x_p\in\mathbb{V}_p^{\mathrm{II}+};\\ \left(\mathcal{N}_0^{\mathrm{II}-}(y'),g\right), & \text{if } x_p\in\mathbb{V}_p^{\mathrm{II}-}.\end{cases} \tag{97}$$

Proof. We will only give the proof for the case $x_p\in\mathbb{V}_p^{\mathrm{I}+}$. Let $\mathsf{T}_{\mathcal{H}}(x)\simeq\mathcal{X}_{\Gamma_0(p^{n+1})\cdot U^p\cap xU^px^{-1}}$ be the closure of the image of $\mathsf{T}(x)$ in the stack $\mathcal{H}_K$ under the morphism $\mathsf{T}(x)\to\mathcal{X}_U^2$ given in Lemma 9.5.3 (I+). Let $\widehat{\mathsf{T}}_{\mathcal{H}}(x)^{\mathrm{ss}}$ be the base change to $\breve{\mathbb{Z}}_p$ of the completion of $\mathsf{T}_{\mathcal{H}}(x)$ along its supersingular locus. We will prove that

$$\widehat{\mathsf{T}}_{\mathcal{H}}(x)^{\mathrm{ss}} = \bigsqcup_{\substack{y\in H'(\mathbb{Q})_0\backslash B\\ q_B(y)=q_{\mathbb{V}}(x)}} \bigsqcup_{\substack{g\in H'_y(\mathbb{Q})_0\backslash H(\mathbb{A}_f^p)/K^p\\ g^{-1}y\in K^px}} \Theta_{\mathcal{H}}^{-1}\left(\mathrm{st}_{x_0y}^{\mathrm{I}+}\left(\mathcal{N}_0(x_0y)\right),g\right). \tag{98}$$

We prove this by constructing explicit maps between the two sides of (98).

Let S be a $\breve{\mathbb{Z}}_p$-scheme such that p is locally nilpotent on S. Let $(z,g)\in\widehat{\mathcal{H}}^{\mathrm{ss}}_K(S)\times H(\mathbb{A}_f^p)/K$ be an element such that $[(z,g)]\in\widehat{\mathsf{T}}_{\mathcal{H}}(x)^{\mathrm{ss}}(S)$. It gives rise to the following object in $\widehat{\mathcal{H}}^{\mathrm{ss}}_K(S)$:

$$\left(\left(E_1\xrightarrow{\pi_1}E'_1,\overline{\eta_1^p}\right),\left(E_2\xrightarrow{\pi_2}E'_2,\overline{\eta_2^p}\right)\right).$$

There exists four quasi-isogenies $\rho_{E,i}:\mathbb{E}_i\times_{\mathbb{F}}\overline{S}\to E_i\times_S\overline{S}$, $\rho'_{E,i}:\mathbb{E}'_i\times_{\mathbb{F}}\overline{S}\to E'_i\times_S\overline{S}$ $(i=1,2)$ of degrees 1 such that $\pi_i\circ\rho_{E,i}=\rho'_{E,i}\circ f_i$ for $i=1,2$. Since $[(z,g)]\in\widehat{\mathsf{T}}(x)^{\mathrm{ss}}(S)$, there exists a cyclic isogeny $\pi:E_1\to E'_2$ of degree p^{n+1} such that π_1 (resp. π_2) is the first (resp. last) degree p isogeny in the standard decomposition of π. Let $\tilde{y}=\left(\rho'_{E,2}\circ\rho'_{\mathbb{E},2}\right)^{-1}\circ\pi\circ\left(\rho'_{E,1}\circ\rho'_{\mathbb{E},1}\right)\in\mathrm{End}^0(\mathbb{E})\simeq B$ and $y=x_0^{-1}\cdot\tilde{y}$. By (93), we have $q_B(y)=q_{\mathbb{V}}(x)$ and $g^{-1}y=x$. Moreover, the following element

$$\left(\left(E_1[p^\infty]\xrightarrow{\pi_1[p^\infty]}E'_1[p^\infty],(\rho_1,\rho'_1)\right),\left(E_2[p^\infty]\xrightarrow{\pi_2[p^\infty]}E'_2[p^\infty],(\rho_2,\rho'_2)\right)\right)$$

lies in the closed formal subscheme $\mathrm{st}_{x_0y}^{\mathrm{I}+}\left(\mathcal{N}_0(x_0y)\right)\subset\mathcal{N}(x_0)$. The reverse direction can be proved easily, therefore (98) is true.

Let $\widehat{\mathsf{T}}^{\mathrm{bl}}(x)$ be the strict transform of $\widehat{\mathsf{T}}_{\mathcal{H}}(x)$ under the blow up morphism $\mathcal{M}_K\to\mathcal{H}_K$. By the definition of $\widehat{\mathsf{T}}(x)_K$, we have

$$\widehat{\mathsf{T}}(x)_K=\widehat{\mathsf{T}}^{\mathrm{bl}}(x)+\mathrm{Exc}_K. \tag{99}$$

Denote by $\widehat{\mathsf{T}}^{\mathrm{bl}}(x)^{\mathrm{ss}}$ the base change to $\breve{\mathbb{Z}}_p$ of the completion of $\widehat{\mathsf{T}}^{\mathrm{bl}}(x)$ along the supersingular locus. By (98), we have

$$\widehat{\mathsf{T}}^{\mathrm{bl}}(x)^{\mathrm{ss}} = \bigsqcup_{\substack{y\in H'(\mathbb{Q})_0\backslash B\\ q_B(y)=q_{\mathbb{V}}(x)}} \bigsqcup_{\substack{g\in H'_y(\mathbb{Q})_0\backslash H(\mathbb{A}_f^p)/K^p\\ g^{-1}y\in K^px}} \Theta_{\mathcal{M}}^{-1}\left(\widetilde{\mathcal{N}}_0^{\mathrm{I}+}(x_0y),g\right).$$

Combining with (99), the formula in the lemma of the case I+ is true. □

Remark 9.8.4. Both of the summations on the right hand side of (97) are finite. The first summation is finite by the Witt theorem. The second summation is also finite since it finally reduces to the finiteness of the double coset $H'_y(\mathbb{Q})\backslash H'_y(\mathbb{A}_f^p)/\tilde{K}^p$ where $\tilde{K}^p \subset H'_y(\mathbb{A}_f^p)$ is a compact open subgroup.

Corollary 9.8.5. *Let $\widehat{\mathsf{T}}(\phi)_K^{\mathrm{ss}}$ be the base change to $\breve{\mathbb{Z}}_p$ of the completion of $\widehat{\mathsf{T}}(\phi)_K$ along its supersingular locus. Then we have the following identity in $K_0^{\widehat{\mathsf{T}}(\phi)_K^{\mathrm{ss}}}(\widehat{\mathcal{M}}_K^{\mathrm{ss}})_{\mathbb{C}}$:*

$$\widehat{\mathsf{T}}(\phi)_K^{\mathrm{ss}} = \sum_{y \in H'(\mathbb{Q})_0 \backslash B} \sum_{g \in H'_y(\mathbb{Q})_0 \backslash H(\mathbb{A}_f^p)/K^p} \phi^p(g^{-1}y) \cdot \Theta_{\mathcal{M}}^{-1} \begin{cases} (\mathcal{Z}(y), g), & \text{if } \phi_p = 1_{H_0(p)}; \\ (\mathcal{Y}(y) - \mathrm{Exc}_{\mathcal{M}}, g), & \text{if } \phi_p = 1_{H_0(p)^\vee}. \end{cases}$$

Proof. Denote by $\mathbb{V}_p(m)$ the following subset of $H_0(p)$:

$$\mathbb{V}_p(m) = \{x \in \mathbb{V} : \det(x) \in \mathbb{Q}_{>0}^\times, \nu_p(q_{\mathbb{V}}(x)) = m\}.$$

For symbols $* \in \{\mathrm{I}, \mathrm{II}\}$ and $? \in \{+, -\}$, define

$$\mathbb{V}_p^{*?}(m) = \{x \in \mathbb{V}_p(m) : x_p \in \mathbb{V}_p^{*?} \cap H_0(p)\}, \quad \mathbb{V}_p^{*?}(m)^\vee = \{x \in \mathbb{V}_p(m) : x_p \in \mathbb{V}_p^{*?} \cap H_0(p)^\vee\}.$$

Therefore we have

$$\widehat{\mathsf{T}}(\phi)_K = \sum_{x \in \widetilde{K} \backslash \mathbb{V}} \phi(x) \cdot \widehat{\mathsf{T}}(x)_K = \sum_{m=-\infty}^{\infty} \sum_{x \in \widetilde{K} \backslash \mathbb{V}(m)} \phi(x) \cdot \widehat{\mathsf{T}}(x)_K$$

We first consider the case $\phi_p = 1_{H_0(p)}$. By Corollary 9.8.3, we have

$$\widehat{\mathsf{T}}(\phi)_K^{\mathrm{ss}} = \sum_{m=0}^{\infty} \sum_{* \in \{\mathrm{I},\mathrm{II}\}} \sum_{? \in \{+,-\}} \sum_{x \in \widetilde{K} \backslash \mathbb{V}_p^{*?}(m)} \phi^p(x^p) \cdot \widehat{\mathsf{T}}(x)_K^{\mathrm{ss}}.$$

For $m = 0$, we have $* = \mathrm{I}$ and $? = +$, hence

$$\begin{aligned} \sum_{* \in \{\mathrm{I},\mathrm{II}\}} \sum_{? \in \{+,-\}} \sum_{x \in \widetilde{K} \backslash \mathbb{V}_p^{*?}(0)} \phi^p(x^p) \cdot \widehat{\mathsf{T}}(x)_K^{\mathrm{ss}} &= \sum_{x \in \widetilde{K} \backslash \mathbb{V}_p^{\mathrm{I}+}(0)} \phi^p(x^p) \cdot \widehat{\mathsf{T}}(x)_K^{\mathrm{ss}} \\ &= \sum_{x \in \widetilde{K} \backslash \mathbb{V}_p^{\mathrm{I}+}(0)} \sum_{\substack{y \in H'(\mathbb{Q})_0 \backslash B \\ q_B(y) = q_{\mathbb{V}}(x)}} \sum_{\substack{g \in H'_y(\mathbb{Q})_0 \backslash H(\mathbb{A}_f^p)/K^p \\ g^{-1}y \in K^p x}} \phi^p(g^{-1}y) \cdot \Theta_{\mathcal{M}}^{-1} \left(\mathrm{Exc}_{\mathcal{M}} + \mathcal{N}_0^{\mathrm{I}+}(x_0 y), g\right) \\ &\overset{\text{Lemma 5.5.2}}{=} \sum_{\substack{y \in H'(\mathbb{Q})_0 \backslash B \\ \nu_p(q_B(y)) = 0}} \sum_{g \in H'_y(\mathbb{Q})_0 \backslash H(\mathbb{A}_f^p)/K^p} \phi^p(g^{-1}y) \cdot \Theta_{\mathcal{M}}^{-1}\left(\mathcal{Z}(y), g\right). \end{aligned}$$

Notice that $\mathcal{D}(y) = \mathcal{Z}(y)$ if $\nu_p(q_B(y)) = 0$ or 1. Similar arguments apply to $m \geqslant 1$, we get (using Lemma 5.5.2 repeatedly)

$$\sum_{* \in \{\mathrm{I},\mathrm{II}\}} \sum_{? \in \{+,-\}} \sum_{x \in \widetilde{K} \backslash \mathbb{V}_p^{*?}(m)} \phi^p(x^p) \cdot \widehat{\mathsf{T}}(x)_K^{\mathrm{ss}} = \sum_{\substack{y \in H'(\mathbb{Q})_0 \backslash B \\ \nu_p(q_B(y)) = m}} \sum_{g \in H'_y(\mathbb{Q})_0 \backslash H(\mathbb{A}_f^p)/K^p} \phi^p(g^{-1}y) \cdot \Theta_{\mathcal{M}}^{-1}\left(\mathcal{D}(y), g\right).$$

Therefore

$$\begin{aligned}\widehat{\mathsf{T}}(\phi)_K^{\mathrm{ss}} &= \sum_{m=0}^{\infty} \sum_{\substack{y\in H'(\mathbb{Q})_0\backslash B\\ \nu_p(q_B(y))=m}} \sum_{g\in H'_y(\mathbb{Q})_0\backslash H(\mathbb{A}_f^p)/K^p} \phi^p(g^{-1}y)\cdot \Theta_{\mathcal{M}}^{-1}\left(\mathcal{D}(y), g\right)\\ &= \sum_{y\in H'(\mathbb{Q})_0\backslash B} \sum_{g\in H'_y(\mathbb{Q})_0\backslash H(\mathbb{A}_f^p)/K^p} \phi^p(g^{-1}y)\cdot \Theta_{\mathcal{M}}^{-1}\left(\sum_{m=0}^{\infty}\mathcal{D}(p^{-m}y), g\right)\\ &= \sum_{y\in H'(\mathbb{Q})_0\backslash B} \sum_{g\in H'_y(\mathbb{Q})_0\backslash H(\mathbb{A}_f^p)/K^p} \phi^p(g^{-1}y)\cdot \Theta_{\mathcal{M}}^{-1}\left(\mathcal{Z}(y), g\right).\end{aligned}$$

Using (36), (37) and (54), the proof of the case $\phi_p = 1_{H_0(p)^\vee}$ is similar. So we omit it. □

Remark 9.8.6. The first sum in the corollary is infinite, so its convergence is not a prior clear. However, the "Fourier coefficients" on the right-hand side—i.e., the sums over vectors with fixed norm $m = q_B(y)$—are finite. In Theorems 9.10.1 and 9.11.2, we compute the nonsingular intersections of these Fourier coefficients and show that their total sum converges to a finite number.

9.9. **Whittaker functions and Eisenstein series.** Let $\nu : \mathrm{GSp}_6 \to \mathbb{G}_m$ be the homomorphism such that $\ker(\nu) = \mathrm{Sp}_6$. Let P be the following Siegel parabolic subgroup of GSp_6:

$$P = \left\{ \begin{pmatrix} a & * \\ 0 & \nu\cdot {}^t a^{-1}\end{pmatrix} \in \mathrm{GSp}_6 \;\middle|\; a\in \mathrm{GL}_3,\ \nu\in\mathbb{G}_m \right\}.$$

Let $M, N \subset P$ be the following groups,

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & \nu\cdot {}^t a^{-1}\end{pmatrix} \in \mathrm{GSp}_6 \;\middle|\; a\in \mathrm{GL}_3,\ \nu\in\mathbb{G}_m \right\},$$

$$N = \left\{ n(b) = \begin{pmatrix} \mathbf{1}_3 & b \\ 0 & \mathbf{1}_3\end{pmatrix} \in \mathrm{GSp}_6 \;\middle|\; b\in \mathrm{Sym}_3 \right\}.$$

For a complex number $s \in \mathbb{C}$, define the following character λ_s of $P(\mathbb{A})$:

$$\lambda_s\left(\begin{pmatrix} a & * \\ 0 & \nu\cdot {}^t a^{-1}\end{pmatrix}\right) = |\nu|_{\mathbb{A}}^{-3s}|\det(a)|_{\mathbb{A}}^{2s}.$$

Let $I(s) = \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GSp}_6(\mathbb{A})}\lambda_s$ be the degenerate principal series of $\mathrm{GSp}_6(\mathbb{A})$. It's easy to see that $I(s) = \otimes'_v I_v(s)$ is the restricted tensor product of the local function space $I_v(s)$.

There exists a Weil representation r of the group $\mathrm{Sp}_6(\mathbb{A})$ on the Schwartz function space $\mathscr{S}(\mathbb{V}^3)$ (cf. [YZZ23, §2.1], see also [Li23, Definition 2.2.1]). For an element $a \in \mathbb{A}^\times$, let

$$d(a) = \begin{pmatrix} \mathbf{1}_3 & 0 \\ 0 & a\cdot\mathbf{1}_3\end{pmatrix}.$$

Let $\Phi \in \mathscr{S}(\mathbb{V}^3)$ be a Schwartz function. Define

$$f_\Phi(g,0) = |\nu(g)|_{\mathbb{A}}^{-3} r\left(d(\nu(g))^{-1}g\right)\Phi(0). \tag{100}$$

The function $f_\Phi(g,0)$ is an element of $I(s)$. We extend it to a standard section $f_\Phi(g,s)$ of $I(s)$. It satisfies

$$f_\Phi(d(\nu)n(b)m(a)g,s) = |v|_{\mathbb{A}}^{-3s-3}|\det(a)|_{\mathbb{A}}^{2s+2} f_\Phi(g,s). \tag{101}$$

Notice that for a single place v and a Schwartz function $\Phi_v \in \mathscr{S}(\mathbb{V}_v^3)$, we can define a section $f_{\Phi_v}(g_v,s) \in I_v(s)$ using similar formulas as (100) and (101).

Define the Siegel Eisenstein series associated to the function $\Phi \in \mathscr{S}(\mathbb{V}^3)$ to be

$$E(g,s,\Phi) = \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{GSp}_6(\mathbb{Q})} f_\Phi(\gamma g,s).$$

This summation is absolutely convergent when $\mathrm{re}(s) > 2$. It extends to a meromorphic function of $s \in \mathbb{C}$ and holomorphic at $s=0$ (cf. [Kud97, Theorem 2.2]). In the following, we will always assume that $\Phi = \Phi_\infty \otimes \Phi_f$ where Φ_∞ is the standard Gaussian function on $\mathbb{V}_\infty^3$, i.e., $\Phi_\infty(\boldsymbol{x}) = e^{-2\pi \mathrm{tr}(q_\infty(\boldsymbol{x}))}$ and $\Phi_f \in \mathbb{V}$.

Let $\psi = \otimes_v \psi_v : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ be the standard character. For $T \in \mathrm{Sym}_3(\mathbb{Q})$, define its T-th Fourier coefficients to be

$$E_T(g,s,\Phi) = \int_{\mathrm{Sym}_3(\mathbb{Q})\backslash \mathrm{Sym}_3(\mathbb{A})} E(n(b)g,s,\Phi)\psi(-Tb)\mathrm{d}b.$$

When $\Phi = \otimes_v \Phi_v$ is decomposable and T is non-singular, we have the following decomposition

$$E_T(g,s,\Phi) = \prod_v W_{T,v}(g_v,s,\Phi_v),$$

where the local Whittaker function is given by

$$W_{T,v}(g_v,s,\Phi_v) = \int_{\mathrm{Sym}_3(\mathbb{Q}_v)} f_{\Phi_v}\left(w_3^{-1}n(b)g_v,s\right)\psi_v\left(-\mathrm{tr}\left(\frac{1}{2}Tb\right)\right)\mathrm{d}b,\ w_3 = \begin{pmatrix} 0 & \mathbf{1}_3 \\ -\mathbf{1}_3 & 0 \end{pmatrix}.$$

Let $\mathbf{1} = (\mathbf{1}_v) \in \mathrm{GSp}(\mathbb{A})$ be the identity element. Kudla [Kud97, Proposition A.6] proved the following:

Lemma 9.9.1. *Let v be a finite prime. Let $\Lambda \subset \mathbb{V}_v$ be a $\mathbb{Z}_v$-lattice of rank 4. Let L be a non-degenerate quadratic $\mathbb{Z}_v$-lattice of rank 3 with fundamental matrix T. Then for positive integers k,*

$$W_{T,v}(1_v,k,1_{\Lambda^3}) = |\det(S)|_p^{3/2} \cdot \mathrm{Den}(\Lambda \oplus H_{2k}^+, L),$$

where S is the a fundamental matrix of the lattice Λ.

For a nonsingular matrix $T \in \mathrm{Sym}_3(\mathbb{Q})$, define

$$\mathrm{Diff}(T) = \{v : T \text{ is not represented by the quadratic space } \mathbb{V}_v\}. \tag{102}$$

The set $\mathrm{Diff}(T)$ has odd cardinality. By [Kud97, Proposition A.4], we also have

$$W_{T,v}(g_v,0,\Phi_v) = 0, \text{ if } v \in \mathrm{Diff}(T).$$

For a finite place p and a Schwartz function $\Phi \in \mathscr{S}(\mathbb{V}^3)$, define

$$E'_p(g,0,\Phi) = \sum_{T:\mathrm{Diff}(T)=\{p\}} E'_T(g,0,\Phi).$$

9.10. **Arithmetic intersection of Hecke correspondences.** Recall that N is an odd and square-free positive integer. Take $U = \Gamma_0(N)(\widehat{\mathbb{Z}})$. Let $\mathcal{X}_0(N)$ be the proper regular integral model of $X_0(N)$ constructed by Katz, Mazur [KM85] and Česnavičius [Č17]. Let $\mathcal{M}_0(N)$ be the blow up of the Deligne–Mumford stack $\mathcal{X}_0(N) \times_{\mathbb{Z}} \mathcal{X}_0(N)$ along its supersingular points with residue field characteristic $p|N$.

Let $H_0(N)$ be the following rank 4 quadratic lattice over $\mathbb{Z}$:

$$H_0(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}.$$

Then $\mathbf{1}_{H_0(N)\otimes\widehat{\mathbb{Z}}} := \otimes_{p<\infty} \mathbf{1}_{H_0(N)\otimes\mathbb{Z}_p}$ is a function in $\mathscr{S}(\mathbb{V}_f)$. For an integer $m > 0$, define

$$\widehat{\mathsf{T}}(m) = \sum_{\substack{x \in \widetilde{K}\backslash \mathbb{V} \\ q_{\mathbb{V}}(x) = m}} \mathbf{1}_{H_0(N)\otimes\widehat{\mathbb{Z}}}(x) \cdot \widehat{\mathsf{T}}(x)_K = \sum_{\substack{x \in \widetilde{K}\backslash H_0(N)\otimes\widehat{\mathbb{Z}} \\ q_{\mathbb{V}}(x) = m}} \widehat{\mathsf{T}}(x)_K. \tag{103}$$

It is a divisor on $\mathcal{M}_0(N)$. For three positive integers m_1, m_2, m_3, define the intersection numbers of three divisors to be

$$\left(\widehat{\mathsf{T}}(m_1) \cdot \widehat{\mathsf{T}}(m_2) \cdot \widehat{\mathsf{T}}(m_3)\right) = \chi\left(\mathcal{M}_0(N), \mathcal{O}_{\widehat{\mathsf{T}}(m_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}_0(N)}} \mathcal{O}_{\widehat{\mathsf{T}}(m_2)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}_0(N)}} \mathcal{O}_{\widehat{\mathsf{T}}(m_3)}\right). \tag{104}$$

Notice that the intersection number may not be finite. The next theorem gives an example where this number is well-defined and finite.

Theorem 9.10.1. *Let m_1, m_2, m_3 be three positive integers such that there is no positive definite binary quadratic form over $\mathbb{Z}$ which represents the three integers m_1, m_2, m_3. Then*

$$\left(\widehat{\mathsf{T}}(m_1) \cdot \widehat{\mathsf{T}}(m_2) \cdot \widehat{\mathsf{T}}(m_3)\right) = -2 \cdot \sum_T E'_T(\mathbf{1}, 0, \Phi_\infty \otimes \mathbf{1}_{(H_0(N)\otimes\widehat{\mathbb{Z}})^3}),$$

where the summation ranges over all the half-integral symmetric positive definite 3×3 matrices T with diagonal elements m_1, m_2, m_3.

Proof. The condition that there is no positive definite binary quadratic form over $\mathbb{Z}$ which represents the three integers m_1, m_2, m_3 implies that the three divisors $\{\widehat{\mathsf{T}}(m_i)\}_{i=1}^3$ have no self-intersections on the generic fiber of $\mathcal{M}_0(N)$ [GK93, Proposition 3.2]. For a prime number p (not necessarily odd), let $B(p)$ be the unique quaternion algebra over $\mathbb{Q}$ which ramifies at p and ∞. Let $\mathcal{M}_{(p)}$ be following formal scheme:

$$\mathcal{M}_{(p)} = \begin{cases} \mathcal{N} \simeq \operatorname{Spf} W[[t_1, t_2]], & \text{if } p \nmid N; \\ \mathcal{M}, & \text{if } p \mid N. \end{cases}$$

Take $\Phi = \Phi_\infty \otimes \mathbf{1}_{\left(H_0(N)\otimes\widehat{\mathbb{Z}}\right)^3} \in \mathscr{S}(\mathbb{V}^3)$, we have the following by Corollary 9.8.5 and (103)

(105)

$$\left(\widehat{\mathsf{T}}(m_1)\cdot\widehat{\mathsf{T}}(m_2)\cdot\widehat{\mathsf{T}}(m_3)\right) = \sum_{p<\infty}\chi\left(\mathcal{M}_0(N)_{(p)}, \mathcal{O}_{\widehat{\mathsf{T}}(m_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}_0(N)_{(p)}}}\mathcal{O}_{\widehat{\mathsf{T}}(m_2)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}_0(N)_{(p)}}}\mathcal{O}_{\widehat{\mathsf{T}}(m_3)}\right)$$
$$= \sum_{p<\infty}\sum_{\substack{\boldsymbol{y}\in H'(\mathbb{Q})_0\backslash B(p)^3\\ q_{B(p)}(y_i)=m_i}}\sum_{g\in H'_{\boldsymbol{y}}(\mathbb{Q})_0\backslash H(\mathbb{A}_f^p)/K^p}\Phi^p(g^{-1}\boldsymbol{y})\cdot\chi\left(\mathcal{M}_{(p)},[^{\mathbb{L}}\otimes_{i=1}^3\mathcal{O}_{\mathcal{Z}(y_i)}]\right)\cdot\log(p)$$
$$= \sum_{p<\infty}\sum_{T}\sum_{\substack{\boldsymbol{y}\in H'(\mathbb{Q})_0\backslash B(p)^3\\ T(\boldsymbol{y})=T}}\sum_{g\in H'_{\boldsymbol{y}}(\mathbb{Q})_0\backslash H(\mathbb{A}_f^p)/K^p}\Phi^p(g^{-1}\boldsymbol{y})\cdot\chi\left(\mathcal{M}_{(p)},[^{\mathbb{L}}\otimes_{i=1}^3\mathcal{O}_{\mathcal{Z}(y_i)}]\right)\cdot\log(p).$$

where the summation symbol for T in the last line ranges over all the half-integral symmetric positive definite 3×3 matrices T with diagonal elements m_1, m_2, m_3.

For $p\nmid N$, [YZZ23, Theorem 5.4.3] shows that

(106)
$$\sum_{\substack{\boldsymbol{y}\in H'(\mathbb{Q})_0\backslash B(p)^3\\ T(\boldsymbol{y})=T}}\sum_{g\in H'_{\boldsymbol{y}}(\mathbb{Q})_0\backslash H(\mathbb{A}_f^p)/K^p}\Phi^p(g^{-1}\boldsymbol{y})\cdot\chi\left(\mathcal{M}_{(p)},[^{\mathbb{L}}\otimes_{i=1}^3\mathcal{O}_{\mathcal{Z}(y_i)}]\right)\cdot\log(p)$$
$$= -2\cdot E'_T(\mathbf{1},0,\Phi_\infty\otimes\mathbf{1}_{\left(H_0(N)\otimes\widehat{\mathbb{Z}}\right)^3}).$$

For $p\,|\,N$, we obtain the same formula as (106) by combining the volume calculation (110) and the intersection of $\mathcal{Z}$-cycles in Theorem 5.6.7 (we refer to the proof of Theorem 9.11.2 in the next section for more details). Then the theorem follows by combining (105) and (106). ☐

Remark 9.10.2. For $N=1$, this theorem is proved by Gross and Keating [GK93, (1.19)].

9.11. **Triple product formula: the minimal ramification case.** Let p be an odd prime number. Let $U = \Gamma_0(p)\cdot U^p$, where $U^p \subset \mathrm{GL}_2(\mathbb{A}_f^p)$ is a sufficiently small compact open subgroup. Let $K = U\times_{\mathbb{G}_m} U = K_p\cdot K^p$. For $i=1,2,3$, let $\phi_i = \phi_{i,p}\otimes\phi_i^p \in \mathscr{S}(\mathbb{V})$ be three Schwartz functions such that

(1) $\phi_{i,\infty}$ is the standard Gaussian function on $\mathbb{V}_\infty$.
(2) $\phi_{i,p} = 1_{H_0(p)}$ or $1_{H_0(p)^\vee}$ and ϕ_i^p is invariant under the group K^p.
(3) There exists a finite place v prime to p such that the Schwartz function $\phi_v = \phi_{1,v}\otimes\phi_{2,v}\otimes\phi_{3,v} \in \mathscr{S}(\mathbb{V}_v^3)$ is regularly supported in the sense of [YZZ23, Definition 4.4.1].

Notice that by condition (3), the three cycles $\widehat{\mathsf{T}}(g_i,\phi_i)$ have no self-intersections on the generic fiber of $\mathcal{M}_{K,(p)}$ because the existence of an intersection point on the generic fiber implies that the function $\phi_1\otimes\phi_2\otimes\phi_3$ is nonzero at some point $\boldsymbol{x}\in\mathbb{V}$ such that the inner product matrix $q(\boldsymbol{x})$ is singular (we refer the readers to [YZZ23, Theorem 5.4.3]). The existence of these functions with some additional non-vanishing properties were proved by Liu [Liu24]. Let $\mathbb{G} = \mathrm{GL}_2\times_{\mathbb{G}_m}\mathrm{GL}_2\times_{\mathbb{G}_m}\mathrm{GL}_2$.

Lemma 9.11.1. *For an element $g=(g_1,g_2,g_3)\in\mathbb{G}(\mathbb{A})$ such that $g_p = \mathbf{1}_p$ is the identity element, let $z\in\widehat{\mathsf{T}}(g_1,\phi_1)\cap\widehat{\mathsf{T}}(g_2,\phi_2)\cap\widehat{\mathsf{T}}(g_3,\phi_3)(\mathbb{F})$ (cf. Definition 9.8.1). Then for $i=1,2,3$, we have*

$$z\in\widehat{\mathsf{T}}(g_i,\phi_i)^{\mathrm{ss}}(\mathbb{F}).$$

Proof. Let $\left(\left(E_1 \xrightarrow{\pi_1} E_1', \overline{(\eta_1^p, \eta_1'^p)}\right), \left(E_2 \xrightarrow{\pi_2} E_2', \overline{(\eta_2^p, \eta_2'^p)}\right)\right)$ be the pair of degree p isogenies corresponding to the point z. Condition (3) implies that $\dim_{\mathbb{Q}} \mathrm{Hom}(E_1, E_2) \geqslant 3$. Therefore E_1 and E_2 must be supersingular elliptic curves. Hence $z \in \widehat{\mathsf{T}}(g_i, \phi_i)^{\mathrm{ss}}(\mathbb{F})$ for $i = 1, 2, 3$. □

Denote by $\Phi \in \mathscr{S}(\mathbb{V}^3)$ by $\Phi = \phi_1 \otimes \phi_2 \otimes \phi_3$. For an element $g = (g_1, g_2, g_3) \in \mathbb{G}(\mathbb{A})$, define $[\mathcal{O}_{\widehat{\mathsf{T}}(g,\Phi)}]$ to be the image of ${}^{\mathbb{L}}\otimes_{i=1}^{3} \mathcal{O}_{\widehat{\mathsf{T}}(g_i,\phi_i)}$ in $\mathrm{Gr}^3 K_0'(\mathcal{M}_{K,(p)})$. It is a formal sum by (96). Define

$$\left(\widehat{\mathsf{T}}(g_1, \phi_1) \cdot \widehat{\mathsf{T}}(g_2, \phi_2) \cdot \widehat{\mathsf{T}}(g_3, \phi_3)\right)_p := \chi\left(\mathcal{M}_{K,(p)}, [\mathcal{O}_{\widehat{\mathsf{T}}(g,\Phi)}]\right) \cdot \log(p).$$

The right hand side is also a formal sum, and we will show that this formal sum converges to a finite number.

Theorem 9.11.2. *Let $\phi_i \in \mathscr{S}(\mathbb{V})$ be three Schwartz functions satisfying (1), (2) and (3). Then for all elements $g = (g_1, g_2, g_3) \in \mathbb{G}(\mathbb{A})$ such that $g_v = \mathbf{1}_v$ (where v is another prime such that (3) holds), we have*

(107)
$$\left(\widehat{\mathsf{T}}(g_1, \phi_1) \cdot \widehat{\mathsf{T}}(g_2, \phi_2) \cdot \widehat{\mathsf{T}}(g_3, \phi_3)\right)_p = -2E_p'(g, 0, \Phi), \quad \text{if } g_p = \mathbf{1}_p \text{ and } \Phi_p = 1_{H_0(p)^3} \text{ or } 1_{H_0(p)^{\vee 3}}.$$

Proof. We first consider the case $\Phi_p = 1_{H_0(p)^3}$. Let $z \in \widehat{\mathsf{T}}(g_i, \phi_i)^{\mathrm{ss}}(\mathbb{F})$, there exists three linearly independent vectors $y_1, y_2, y_3 \in B$ such that $z \in \bigcap_{i=1}^{3} \mathcal{Z}(y_i)$. Let T be the fundamental matrix of the quadratic lattice L spanned by y_1, y_2 and y_3. Then

$$\mathrm{Diff}(T) = \{p\}.$$

Moreover, we have

$$\begin{aligned}\chi\left(\mathcal{M}, [{}^{\mathbb{L}}\otimes_{i=1}^{3} \mathcal{O}_{\mathcal{Z}(y_i)}]\right) &= \mathrm{Int}^{\mathcal{Z}}(L) = \partial\mathrm{Den}(H_0(p), L) \\ &= p^4 (p-1)^{-2} \cdot \log(p)^{-1} \cdot W_{T,p}'(1, 0, 1_{H_0(p)^3}).\end{aligned}$$

By Corollary 9.8.5, the formal sum $[\mathcal{O}_{\widehat{\mathsf{T}}(g,\Phi)}]$ corresponds to the following formal sum in $\mathrm{Gr}^3 K_0'(\widehat{\mathcal{M}}_K^{\mathrm{ss}})$:

$$\sum_{\boldsymbol{y} \in H'(\mathbb{Q})_0 \backslash B^3} \sum_{g \in H_{\boldsymbol{y}}'(\mathbb{Q})_0 \backslash H(\mathbb{A}_f^p)/K^p} \Phi^p(g^{-1}\boldsymbol{y}) \cdot \Theta_{\mathcal{M}}^{-1}\left([{}^{\mathbb{L}}\otimes_{i=1}^{3} \mathcal{O}_{\mathcal{Z}(y_i)}]\right) \tag{108}$$

Therefore

$$\begin{aligned}&\left(\widehat{\mathsf{T}}(g_1, \Phi_1) \cdot \widehat{\mathsf{T}}(g_2, \Phi_2) \cdot \widehat{\mathsf{T}}(g_3, \Phi_3)\right)_p \\ &= \sum_{\boldsymbol{y} \in H'(\mathbb{Q})_0 \backslash B^3} \sum_{g \in H_{\boldsymbol{y}}'(\mathbb{Q})_0 \backslash H(\mathbb{A}_f^p)/K^p} \Phi^p(g^{-1}\boldsymbol{y}) \cdot \chi\left(\mathcal{M}, [{}^{\mathbb{L}}\otimes_{i=1}^{3} \mathcal{O}_{\mathcal{Z}(y_i)}]\right) \cdot \log(p) \\ &= \sum_{T: \mathrm{Diff}(T) = \{p\}} \sum_{\substack{\boldsymbol{y} \in H'(\mathbb{Q})_0 \backslash B^3 \\ T(\boldsymbol{y}) = T}} \sum_{g \in H_{\boldsymbol{y}}'(\mathbb{Q})_0 \backslash H(\mathbb{A}_f^p)/K^p} \Phi^p(g^{-1}\boldsymbol{y}) \cdot \frac{p^4}{(p-1)^2} \cdot W_{T,p}'(1, 0, 1_{H_0(p)^3}) \\ &= \frac{p^4}{(p-1)^2} \cdot \frac{1}{\mathrm{vol}(K^p)} \cdot \int_{\mathrm{SO}(B)(\mathbb{A}_f^p)} \Phi^p(g^{-1}\boldsymbol{y}) \mathrm{d}g \cdot W_{T,p}'(1, 0, 1_{H_0(p)^3}).\end{aligned} \tag{109}$$

We use the same measures for orthogonal groups as fixed in [YZZ23, §1.4]. By the calculations of the volume factor in Theorem 5.4.3 of *loc.cit.* and Siegel–Weil formula, we have

$$\frac{p^4}{(p-1)^2}\cdot\frac{1}{\mathrm{vol}(K^p)}\cdot\int_{\mathrm{SO}(B)(\mathbb{A}_f^p)}\Phi^p(g^{-1}\boldsymbol{y})\mathrm{d}g=-2\cdot\prod_{v\neq p}W_{T,v}(g_v,0,\Phi_v). \tag{110}$$

Therefore (107) is true when $g_p=\mathbf{1}_p$.

Now we consider the case $\Phi_p=1_{H_0(p)^{\vee 3}}$. Let $z\in\widehat{\mathsf{T}}(g_i,\Phi_i)^{\mathrm{ss}}(\mathbb{F})$, there exists three linearly independent vectors $y_1,y_2,y_3\in B$ such that $z\in\bigcap_{i=1}^{3}(\mathcal{Y}(y_i)-\mathrm{Exc}_{\mathcal{M}})$. Let T be the fundamental matrix of the quadratic lattice L spanned by y_1,y_2 and y_3. Then

$$\mathrm{Diff}(T)=\{p\}.$$

Moreover, we have

$$\begin{aligned}\chi\left(\mathcal{M},[{}^{\mathbb{L}}\otimes_{i=1}^{3}\mathcal{O}_{\mathcal{Y}(y_i)-\mathrm{Exc}_{\mathcal{M}}}]\right)&=\mathrm{Int}^{\mathcal{Y}}(L)+1=\partial\mathrm{Den}(H_0(p)^\vee,L)\\&=p^4(p-1)^{-2}\cdot\log(p)^{-1}\cdot W'_{T,p}(1,0,1_{H_0(p)^{\vee 3}}).\end{aligned}$$

By Lemma 9.8.5, we have

$$[\mathcal{O}_{\widehat{\mathsf{T}}^{\mathrm{ss}}(g,\Phi)}]=\sum_{\boldsymbol{y}\in H'(\mathbb{Q})_0\backslash B^3}\ \sum_{g\in H'_{\boldsymbol{y}}(\mathbb{Q})_0\backslash H(\mathbb{A}_f^p)/K^p}\Phi^p(g^{-1}\boldsymbol{y})\cdot\Theta_{\mathcal{M}}^{-1}\left([{}^{\mathbb{L}}\otimes_{i=1}^{3}\mathcal{O}_{\mathcal{Y}(y_i)-\mathrm{Exc}_{\mathcal{M}}}]\right).$$

The remaining parts are similar to the proof of the case $\Phi_p=1_{H_0(p)^3}$. □

9.12. **(Semi-global) Arithmetic Siegel–Weil formula on $\mathcal{M}_K$.** In this section, we prove a (semi-global) Arithmetic Siegel–Weil formula following the idea in [LZ22b, §12]. We keep the notations fixed in §9.7 and §9.8. Fix an odd prime p, let $U=\Gamma_0(p)\cdot U^p$ where $U^p\subset\mathrm{GL}_2(\mathbb{A}_f^p)$ is a sufficiently small compact open subgroup and $K=U\times_{\mathbb{G}_m}U$. Define $\mathcal{M}_K^\circ=\mathcal{M}_K\times_{\mathcal{H}_K}\mathcal{H}_K^\circ$.

We say a Schwartz function

$$\varphi_K=\bigotimes_{v\nmid\infty}\varphi_{K,v}\in\mathscr{S}(\mathbb{V}_f^r)$$

is p-admissible if it is K-invariant and $\varphi_{K,p}=\mathbf{1}_{H_0(p)^r}$. First, we consider a special p-admissible Schwartz function of the form

$$\varphi_K=(\varphi_i)\in\mathscr{S}(\mathbb{V}_f^r),\ \ \varphi_i=\mathbf{1}_{\Omega_i},\ \ i=1,\cdots,r.$$

where $\Omega_i\subset\mathbb{V}_f$ is a K-invariant open compact subset such that $\Omega_{i,p}=H_0(p)$.

For a nonsingular matrix $T\in\mathrm{Sym}_r(\mathbb{Q})$, denote by $\mathcal{Z}_{\mathcal{H}}^\circ(T,\varphi_K)$ a stack whose fiber category over a connected scheme S consists of the following objects,

$$\left(\left(\left(E_1\xrightarrow{\pi_1}E_1'\right),\left(E_2\xrightarrow{\pi_2}E_2'\right),\overline{(\eta_1^p,\eta_2^p)}\right),(x_1,\cdots,x_r)\right)$$

where $\left(\left(E_1\xrightarrow{\pi_1}E_1'\right),\left(E_2\xrightarrow{\pi_2}E_2'\right),\overline{(\eta_1^p,\eta_2^p)}\right)$ is an element in $\mathcal{H}_K^\circ(S)$ (see (90)). For $1\leqslant i\leqslant 3$, the element $x_i\in\mathrm{Hom}_S(E_1,E_2)$ is an isogeny such that the quasi-isogeny $\pi_2\circ x_i\circ\pi_1^{-1}:E_1'\to E_2'$ is also an isogeny with the following restrictions: Under the standard basis $e_1=(1,0),e_2=(0,1)$ of $\left(\widehat{\mathbb{Z}}^p\right)^2$, the morphism $\eta_2^p\circ T^p(x_i)\circ(\eta_1^p)^{-1}:\left(\widehat{\mathbb{Z}}^p\right)^2\to\left(\widehat{\mathbb{Z}}^p\right)^2$ can be identified with a 2×2 matrix, or just an element in $\mathbb{V}_f^p$. The isogenies x_i should satisfy

$$\eta_2^p \circ T^p(x_i) \circ (\eta_1^p)^{-1} \in \Omega_i^{(p)}, \quad 1 \leqslant i \leqslant r.$$

The natural morphism $\mathcal{Z}_{\mathcal{H}}^{\circ}(T, \varphi_K) \to \mathcal{H}_K^{\circ}$ is finite and unramified, and thus it defines a cycle on $\mathcal{H}_K^{\circ}$ which we still denote as $\mathcal{Z}_{\mathcal{H}}^{\circ}(T, \varphi_K)$. Define $\mathcal{Z}_{\mathcal{H}}(T, \varphi_K)$ to be the Zariski closure of $\mathcal{Z}_{\mathcal{H}}^{\circ}(T, \varphi_K)$ in $\mathcal{H}_K$, and

$$\mathcal{Z}(T, \varphi_K) := \mathcal{Z}_{\mathcal{H}}(T, \varphi_K) \times_{\mathcal{H}_K} \mathcal{M}_K.$$

For a general p-admissible Schwartz function $\varphi_K \in \mathscr{S}(\mathbb{V}_f^r)$ (which can be written as a $\mathbb{C}$-linear combination of special p-admissible functions, after possibly shrinking K), we obtain a cycle $\mathcal{Z}(T, \varphi_K) \in \mathrm{Z}^*(\mathcal{M}_K)_{\mathbb{C}}$ by extending $\mathbb{C}$-linearly.

For $r = 3$, if $\mathcal{Z}_{\mathcal{H}}(T, \varphi_K)$ is nonempty, we have $\mathcal{Z}_{\mathcal{H}}^{\circ}(T, \varphi_K)(\overline{\mathbb{F}}_p) \neq \varnothing$. By the proof of Lemma 9.11.1 and Theorem 9.11.2, the morphism $\mathcal{Z}_{\mathcal{H}}^{\circ}(T, \varphi_K) \to \mathcal{H}_K^{\circ}$ sends $|\mathcal{Z}_{\mathcal{H}}^{\circ}(T, \varphi_K)(\overline{\mathbb{F}}_p)|$ to the supersingular locus of $\mathcal{H}_K^{\circ}$ modulo p, and we have $\mathrm{Diff}(T) = \{p\}$. Moreover, the cycle $\mathcal{Z}_{\mathcal{H}}^{\circ}(T, \varphi_K)$ itself is finite over $\mathbb{F}_p$, hence $\mathcal{Z}_{\mathcal{H}}^{\circ}(T, \varphi_K) = \mathcal{Z}_{\mathcal{H}}(T, \varphi_K)$.

Recall that we use $\widehat{\mathcal{M}}_K^{\mathrm{ss}}$ to denote the base change to $\breve{\mathbb{Z}}_p$ of the formal completion of $\mathcal{M}_K$ along Exc_K. For special p-admissible Schwartz function φ_K, denote by $\hat{\mathcal{Z}}^{\mathrm{ss}}(T, \varphi_K)$ the base change to $\breve{\mathbb{Z}}_p$ of the completion of $\mathcal{Z}(T, \varphi_K)$ along its supersingular locus. For a general p-admissible Schwartz function $\varphi_K \in \mathscr{S}(\mathbb{V}_f^r)$, we obtain a cycle $\hat{\mathcal{Z}}^{\mathrm{ss}}(T, \varphi_K) \in \mathrm{Z}^*(\widehat{\mathcal{M}}_K^{\mathrm{ss}})_{\mathbb{C}}$ by extending $\mathbb{C}$-linearly. We state the following proposition about the p-adic uniformization of the cycle $\mathcal{Z}(T, \varphi_K)$, which originates from the comparison of the moduli problem defining the (semi)-global special cycle $\mathcal{Z}(T, \varphi_K)$ and that of the local special cycle $\mathcal{Z}(L)$ (Definition 4.9.1 and 4.13.1). Other than that, the proof is similar to that of [LZ22b, Proposition 12.7.1], so we omit it.

Proposition 9.12.1. *For $1 \leqslant r \leqslant 3$, assume that $\varphi_K \in \mathscr{S}(\mathbb{V}_f^r)$ is p-admissible. Then for all $T \in \mathrm{Sym}_r(\mathbb{Q})_{>0}$, the p-adic uniformization isomorphism in (95) induces the following identity in $K_0^{\hat{\mathcal{Z}}^{\mathrm{ss}}(T, \varphi_K)}(\widehat{\mathcal{M}}_K^{\mathrm{ss}})$,*

$$\hat{\mathcal{Z}}^{\mathrm{ss}}(T, \varphi_K) = \sum_{\substack{\boldsymbol{x} \in H'(\mathbb{Q})_0 \backslash B^r \\ q_B(\boldsymbol{x}) = T}} \sum_{g \in H'_{\boldsymbol{x}}(\mathbb{Q})_0 \backslash H(\mathbb{A}_f^p)/K^p} \Theta_{\mathcal{M}}^{-1}\left(\mathcal{Z}(\boldsymbol{x}), g\right). \tag{111}$$

Assume $T \in \mathrm{Sym}_3(\mathbb{Q})_{>0}$. Let t_1, t_2, t_3 be the diagonal entries of T. Let $\varphi_K \in \mathscr{S}(\mathbb{V}_f^3)$ be a special p-admissible Schwartz function. Define

$$\mathrm{Int}_{T,p}(\varphi_K) := \chi\left(\mathcal{Z}(T, \varphi_K), \mathcal{O}_{\mathcal{Z}(t_1, \varphi_K)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}_K}} \mathcal{O}_{\mathcal{Z}(t_2, \varphi_K)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}_K}} \mathcal{O}_{\mathcal{Z}(t_3, \varphi_K)}\right) \cdot \log p.$$

We extend the definition of $\mathrm{Int}_{T,p}(\varphi_K)$ to a general p-admissible $\varphi_K \in \mathscr{S}(\mathbb{V}_f^3)$ by extending $\mathbb{C}$-linearly.

Theorem 9.12.2. *Let $\varphi_K \in \mathscr{S}(\mathbb{V}_f^3)$ be a p-admissible Schwartz function. For all $T \in \mathrm{Sym}_3(T)_{>0}$,*

$$\mathrm{Int}_{T,p}(\varphi_K) = -2 \cdot E'_T(\mathbf{1}, 0, \Phi_\infty \otimes \varphi_K). \tag{112}$$

Proof. Since $\mathcal{Z}(T,\varphi_K)$ is supported on the supersingular locus, by (111) we know that

$$\begin{aligned}&\mathrm{Int}_{T,p}(\varphi_K)\\ &=\sum_{\substack{\boldsymbol{x}\in H'(\mathbb{Q})_0\backslash B^3\\ q_B(\boldsymbol{x})=T}}\ \sum_{g\in H'_{\boldsymbol{x}}(\mathbb{Q})_0\backslash H(\mathbb{A}_f^p)/K^p}\varphi_K(g^{-1}\boldsymbol{x})\cdot\chi\left(\mathcal{M},\mathcal{O}_{\mathcal{Z}(x_1)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\mathcal{Z}(x_2)}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{M}}}\mathcal{O}_{\mathcal{Z}(x_3)}\right)\cdot\log p\end{aligned}$$

By (109) and (110), we have $\mathrm{Int}_{T,p}(\varphi_K)=-2\cdot E'_T(\mathbf{1},0,\Phi_\infty\otimes\varphi_K)$. □

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